

# Self-normalized moderate deviation and laws of the iterated logarithm under G-expectation\*

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## Abstract

The sub-linear expectation or called G-expectation is a nonlinear expectation having advantage of modeling non-additive probability problems and the volatility uncertainty in finance. Let  $\{X_n; n \geq 1\}$  be a sequence of independent random variables in a sub-linear expectation space  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ . Denote  $S_n = \sum_{k=1}^n X_k$  and  $V_n^2 = \sum_{k=1}^n X_k^2$ . In this paper, a moderate deviation for self-normalized sums, that is, the asymptotic capacity of the event  $\{S_n/V_n \geq x_n\}$  for  $x_n = o(\sqrt{n})$ , is found both for identically distributed random variables and independent but not necessarily identically distributed random variables. As an applications, the self-normalized laws of the iterated logarithm are obtained.

**Keywords:** non-linear expectation; capacity; self-normalization; law of the iterated logarithm; moderate deviation.

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## 1 Introduction

Let  $\{X_n; n \geq 1\}$  be a sequence of independent and identically distributed random variables on a probability space  $(\Omega, \mathcal{F}, P)$ . Set  $S_n = \sum_{j=1}^n X_j$ ,  $V_n^2 = \sum_{j=1}^n X_j^2$ . The well-known classical Hartman-Wintner law of the iterated logarithm (LIL) says that, if  $EX_1 = 0$  and  $EX_1^2 = \sigma^2 > 0$ , then

$$P \left( \limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2\sigma^2 n \log \log n}} = 1 \right) = 1, \quad (1.1)$$

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and its converse is obtained by Strassen (1966). Griffin and Kuelbs (1989) obtained a self-normalized law of the iterated logarithm under the following condition

$$\lim_{x \rightarrow \infty} \frac{x^2 P(|X_1| \geq x)}{EX_1^2 I\{|X_1| \leq x\}} = 0. \quad (1.2)$$

That is, if  $EX_1 = 0$  and (1.2) is satisfied, then

$$P\left(\limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} = 1\right) = E\left[I\left\{\limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} = 1\right\}\right] = 1. \quad (1.3)$$

On the other hand, Shao (1997)'s self-normalized moderate deviations gives us the asymptotic probability of  $P(S_n \geq x_n V_n)$  as follows. If  $EX_1 = 0$  and (1.2) is satisfied, then for any real sequence  $\{x_n\}$  with  $x_n \rightarrow \infty$  and  $x_n = o(\sqrt{n})$ ,

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln P(S_n \geq x_n V_n) = \lim_{n \rightarrow \infty} x_n^{-2} \ln E[I\{S_n \geq x_n V_n\}] = -\frac{1}{2}. \quad (1.4)$$

The result is closely related to the Cramér (1938) large deviation. It is known [cf. Petrov (1975)] that

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln P\left(\frac{S_n}{\sqrt{n}} \geq x_n\right) = -\frac{1}{2}.$$

holds for any sequence of  $\{x_n\}$  with  $x_n \rightarrow \infty$  and  $x_n = o(\sqrt{n})$  if and only if  $EX_1 = 0$ ,  $EX_1^2 = 1$  and  $E \exp\{t_0 |X_1|\} < \infty$  for some  $t_0 > 0$ . The self-normalized limit theorems put a totally new countenance upon classical limit theorems.

The purpose of this paper is to study the self-normalized moderate deviation and self-normalized law of the iterated logarithm for random variables in a sub-linear expectation space. The sub-linear expectation or called G-expectation is a nonlinear expectation advancing the notions of g-expectations, backward stochastic differential equations and providing a flexible framework to model non-additive probability problems and the volatility uncertainty in finance. Peng (2006, 2008a, 2008b) introduced a general framework of the sub-linear expectation of random variables by relaxing the linear property of the classical linear expectation to the sub-additivity and positive homogeneity (cf. Definition 2.1 below), and introduced the notions of G-normal random variable, G-Brownian motion, independent and identically distributed random variables etc under the sub-linear expectations. The construction of sub-linear expectations on the space of continuous paths and discrete time paths can also be found in Yan et al (2012) and Nutz and Handel (2013). For basic properties of the

sub-linear expectations, one can refer to Peng (2008b,2009,2010a,etc). Under the sub-linear expectation, the central limit theorem was first established by Peng (2008b), large deviations and moderate deviations were derived by Gao and Xu (2011, 2012), and the Hartman-Winter laws of the iterated logarithm were recently established by Chen and Hu (2014) for bounded random variables. Even for bounded random variables in a sub-linear expectation space, the self-normalized laws of the iterated logarithm can not follow from the Hartman-Winter laws of the iterated logarithm directly because  $V_n^2/n$  does not converge to a constant. We will show that (1.3) and (1.4) are also true for random variables in a sub-linear expectation space when the expectation  $E$  being replaced by the sub-linear expectation  $\widehat{\mathbb{E}}$ . The main difficulty for proving the results under the sub-linear expectation is that we can not use the additivity of the probability and the expectation which is essential in the proof of classical results under the classical linear expectation. For example, the simple facts  $-\widehat{\mathbb{E}}[X_1^2 I\{|X_1| \leq x\}] = \widehat{\mathbb{E}}[-X_1^2 I\{|X_1| \leq x\}]$  and

$$\widehat{\mathbb{E}}[X] = \int_0^\infty \widehat{\mathbb{E}}[I\{X \geq x\}] dx + \int_{-\infty}^0 \widehat{\mathbb{E}}[I\{X \leq -x\}] dx$$

are not true now, and the conjugate method [cf. (4.9) of Petrov (1965)] is not available. These are the main keys to establish (1.2) in Shao (1997). Our main results on the self-normalized law of the iterated logarithm and self-normalized moderate deviations for independent and identically distributed random variables will be given in Section 3. In the next section, we state basic settings in a sub-linear expectation space including, capacity, independence, identical distribution etc. One can skip this section if he/she is familiar with these concepts. The proofs are given in Section 4, where a Bernstein's type inequality for the maximum sum of independent random variables is also established for proving the law of the iterated logarithm. In Section 5, we consider the special case of  $G$ -normal random variables. We prove a finer self-normalized law of the iterated logarithm which shows the results are the same under the upper capacity and the lower capacity generated by the sub-linear expectation. In Section 6, we give similar results for independent but not necessarily identically distributed random variables.

## 2 Basic Settings

We use the framework and notations of Peng (2008b). Let  $(\Omega, \mathcal{F})$  be a given measurable space and let  $\mathcal{H}$  be a linear space of real functions defined on  $(\Omega, \mathcal{F})$  such that if  $X_1, \dots, X_n \in \mathcal{H}$  then  $\varphi(X_1, \dots, X_n) \in \mathcal{H}$  for each  $\varphi \in C_b(\mathbb{R}^n) \cup C_{l,Lip}(\mathbb{R}^n)$ , where  $C_b(\mathbb{R}^n)$  denote the space of all bounded continuous functions and  $C_{l,Lip}(\mathbb{R}^n)$  denotes the linear space of (local Lipschitz) functions  $\varphi$  satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

for some  $C > 0, m \in \mathbb{N}$  depending on  $\varphi$ .

$\mathcal{H}$  is considered as a space of “random variables”. In this case we denote  $X \in \mathcal{H}$ . Further, we let  $C_{b,Lip}(\mathbb{R}^n)$  denote the space of all bounded and Lipschitz functions on  $\mathbb{R}^n$ .

### 2.1 Sub-linear expectation and capacity

**Derinition 2.1** *A sub-linear expectation  $\widehat{\mathbb{E}}$  on  $\mathcal{H}$  is a functional  $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \overline{\mathbb{R}}$  satisfying the following properties: for all  $X, Y \in \mathcal{H}$ , we have*

- (a) **Monotonicity:** *If  $X \geq Y$  then  $\widehat{\mathbb{E}}[X] \geq \widehat{\mathbb{E}}[Y]$ ;*
- (b) **Constant preserving:**  $\widehat{\mathbb{E}}[c] = c$ ;
- (c) **Sub-additivity:**  $\widehat{\mathbb{E}}[X + Y] \leq \widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$  *whenever  $\widehat{\mathbb{E}}[X] + \widehat{\mathbb{E}}[Y]$  is not of the form  $+\infty - \infty$  or  $-\infty + \infty$ ;*
- (d) **Positive homogeneity:**  $\widehat{\mathbb{E}}[\lambda X] = \lambda \widehat{\mathbb{E}}[X]$ ,  $\lambda \geq 0$ .

Here  $\overline{\mathbb{R}} = [-\infty, \infty]$ . The triple  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$  is called a sub-linear expectation space. Give a sub-linear expectation  $\widehat{\mathbb{E}}$ , let us denote the conjugate expectation  $\widehat{\mathcal{E}}$  of  $\widehat{\mathbb{E}}$  by

$$\widehat{\mathcal{E}}[X] := -\widehat{\mathbb{E}}[-X], \quad \forall X \in \mathcal{H}.$$

Next, we introduce the capacities corresponding to the sub-linear expectations. Let  $\mathcal{G} \subset \mathcal{F}$ . A function  $V : \mathcal{G} \rightarrow [0, 1]$  is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \quad \text{and} \quad V(A) \leq V(B) \quad \forall A \subset B, \quad A, B \in \mathcal{G}.$$

It is called to be sub-additive if  $V(A \cup B) \leq V(A) + V(B)$  for all  $A, B \in \mathcal{G}$  with  $A \cup B \in \mathcal{G}$ .

Let  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$  be a sub-linear space, and  $\widehat{\mathcal{E}}$  be the conjugate expectation of  $\widehat{\mathbb{E}}$ . It is natural to define the capacity of a set  $A$  to be the sub-linear expectation of the indicator function  $I_A$  of  $A$ . However,  $I_A$  may be not in  $\mathcal{H}$ . So, we denote a pair  $(\mathbb{V}, \mathcal{V})$  of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) := 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F},$$

where  $A^c$  is the complement set of  $A$ . Then  $\mathbb{V}$  is sub-additive and

$$\begin{aligned} \mathbb{V}(A) &= \widehat{\mathbb{E}}[I_A], \quad \mathcal{V}(A) = \widehat{\mathcal{E}}[I_A], \quad \text{if } I_A \in \mathcal{H} \\ \widehat{\mathbb{E}}[f] &\leq \mathbb{V}(A) \leq \widehat{\mathbb{E}}[g], \quad \widehat{\mathcal{E}}[f] \leq \mathcal{V}(A) \leq \widehat{\mathcal{E}}[g], \quad \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}. \end{aligned} \tag{2.1}$$

Further, we define an extension of  $\widehat{\mathbb{E}}^*$  of  $\widehat{\mathbb{E}}$  by

$$\widehat{\mathbb{E}}^*[X] = \inf\{\widehat{\mathbb{E}}[Y] : X \leq Y, Y \in \mathcal{H}\}, \quad \forall X : \Omega \rightarrow \mathbb{R},$$

where  $\inf \emptyset = +\infty$ . Then

$$\begin{aligned} \widehat{\mathbb{E}}^*[X] &= \widehat{\mathbb{E}}[X] \quad \text{if } X \in \mathcal{H}, \quad \mathbb{V}(A) = \widehat{\mathbb{E}}^*[I_A], \\ \widehat{\mathbb{E}}[f] &\leq \widehat{\mathbb{E}}^*[X] \leq \widehat{\mathbb{E}}[g] \quad \text{if } f \leq X \leq g, f, g \in \mathcal{H}. \end{aligned}$$

Also, we define the Choquet integrals/expecations  $(C_{\mathbb{V}}, C_{\mathcal{V}})$  by

$$C_V[X] = \int_0^\infty V(X \geq t) dt + \int_{-\infty}^0 [V(X \geq t) - 1] dt$$

with  $V$  being replaced by  $\mathbb{V}$  and  $\mathcal{V}$  respectively. It can be verified that (c.f. Zhang (2014)), if  $\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0$ , then

$$\widehat{\mathbb{E}}[|X|] \leq C_{\mathbb{V}}(|X|). \tag{2.2}$$

**Derinition 2.2** (I) *A sub-linear expectation  $\widehat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$  is called to be countably sub-additive if it satisfies*

- (e) **Countable sub-additivity:**  $\widehat{\mathbb{E}}[X] \leq \sum_{n=1}^\infty \widehat{\mathbb{E}}[X_n]$ , whenever  $X \leq \sum_{n=1}^\infty X_n$ ,  $X, X_n \in \mathcal{H}$  and  $X \geq 0, X_n \geq 0, n = 1, 2, \dots$ ;

(II) A function  $V : \mathcal{F} \rightarrow [0, 1]$  is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n) \quad \forall A_n \in \mathcal{F}.$$

(III) A capacity  $V : \mathcal{F} \rightarrow [0, 1]$  is called a continuous capacity if it satisfies

(III1) **Ccontinuity from below:**  $V(A_n) \uparrow V(A)$  if  $A_n \uparrow A$ , where  $A_n, A \in \mathcal{F}$ ;

(III2) **Continuity from above:**  $V(A_n) \downarrow V(A)$  if  $A_n \downarrow A$ , where  $A_n, A \in \mathcal{F}$ .

Note that if  $V$  is a countably sub-additive capacity, then

$$0 \leq V\left(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i\right) \leq V\left(\bigcup_{i=n}^{\infty} A_i\right) \leq \sum_{i=n}^{\infty} V(A_i).$$

So, we have the following Borel-Cantelli's Lemma.

**Lemma 2.1** (Borel-Cantelli's Lemma) *Let  $\{A_n, n \geq 1\}$  be a sequence of events in  $\mathcal{F}$ . Suppose that  $V$  is a countably sub-additive capacity. If  $\sum_{n=1}^{\infty} V(A_n) < \infty$ , then*

$$V(A_n \text{ i.o.}) = 0, \quad \text{where } \{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i.$$

## 2.2 Independence and distribution

**Derinition 2.3** (Peng (2006, 2008b))

- (i) **(Identical distribution)** *Let  $\mathbf{X}_1$  and  $\mathbf{X}_2$  be two  $n$ -dimensional random vectors defined respectively in sub-linear expectation spaces  $(\Omega_1, \mathcal{H}_1, \widehat{\mathbb{E}}_1)$  and  $(\Omega_2, \mathcal{H}_2, \widehat{\mathbb{E}}_2)$ . They are called identically distributed, denoted by  $\mathbf{X}_1 \stackrel{d}{=} \mathbf{X}_2$  if*

$$\widehat{\mathbb{E}}_1[\varphi(\mathbf{X}_1)] = \widehat{\mathbb{E}}_2[\varphi(\mathbf{X}_2)], \quad \forall \varphi \in C_{l,Lip}(\mathbb{R}^n),$$

*whenever the sub-expectations are finite. A sequence  $\{X_n; n \geq 1\}$  of random variables is said to be identically distributed if  $X_i \stackrel{d}{=} X_1$  for each  $i \geq 1$ .*

- (ii) **(Independence)** *In a sub-linear expectation space  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$ , a random vector  $\mathbf{Y} = (Y_1, \dots, Y_n)$ ,  $Y_i \in \mathcal{H}$  is said to be independent to another random vector  $\mathbf{X} = (X_1, \dots, X_m)$ ,  $X_i \in \mathcal{H}$  under  $\widehat{\mathbb{E}}$  if for each test function  $\varphi \in C_{l,Lip}(\mathbb{R}^m \times \mathbb{R}^n)$  we have*

$$\widehat{\mathbb{E}}[\varphi(\mathbf{X}, \mathbf{Y})] = \widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(\mathbf{x}, \mathbf{Y})] |_{\mathbf{x}=\mathbf{X}}],$$

*whenever  $\overline{\varphi}(\mathbf{x}) := \widehat{\mathbb{E}}[|\varphi(\mathbf{x}, \mathbf{Y})|] < \infty$  for all  $\mathbf{x}$  and  $\widehat{\mathbb{E}}[|\overline{\varphi}(\mathbf{X})|] < \infty$ .*

- (iii) **(IID random variables)** A sequence of random variables  $\{X_n; n \geq 1\}$  is said to be independent, if  $X_{i+1}$  is independent to  $(X_1, \dots, X_i)$  for each  $i \geq 1$ , and it is said to be identically distributed, if  $X_i \stackrel{d}{=} X_1$  for each  $i \geq 1$ .

### 3 Main results

If  $x \in \mathbb{R}$ ,  $A \subset \mathbb{R}$ , then the distance from  $x$  to  $A$  is defined as

$$d(x, A) = \inf_{y \in A} |x - y|.$$

If  $\{x_n\}$  is a real sequence, then  $C(\{x_n\})$  denotes its cluster set, that is,  $C(\{x_n\}) = \{y : \liminf_{n \rightarrow \infty} |x_n - y| = 0\}$ . We write  $\{x_n\} \rightarrow A$  if both  $\lim_{n \rightarrow \infty} d(x_n, A) = 0$  and  $C(\{x_n\}) = A$ .

Let  $\{X, X_n; n \geq 1\}$  be a sequence of independent and identically distributed random variables in the sub-linear expectation space  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$  with  $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0$ . Denote

$$S_n = X_1 + \dots + X_n, \quad V_n^2 = X_1^2 + \dots + X_n^2.$$

Define  $l(x) = \widehat{\mathbb{E}}[X^2 \wedge x^2]$ .

Our main result is the following self-normalized law of iterated logarithm (LIL).

**Theorem 3.1** *Suppose*

- (I)  $\mathbb{V}(|X| \geq x) = o(x^{-2}l(x))$  as  $x \rightarrow \infty$ ;
- (II)  $\limsup_{x \rightarrow \infty} \frac{\widehat{\mathbb{E}}[X^2 \wedge x^2]}{\widehat{\mathcal{E}}[X^2 \wedge x^2]} < \infty$  (say,  $< r^2 < \infty$ );
- (III)  $\widehat{\mathbb{E}}[(|X| - c)^+] \rightarrow 0$  as  $c \rightarrow \infty$ .

Then

$$\mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{|S_n|}{V_n \sqrt{2 \log \log n}} \leq 1 \right) = 1 \quad (3.1)$$

when  $\mathbb{V}$  is countably sub-additive; and

$$\mathbb{V} \left( \left\{ \frac{S_n}{V_n \sqrt{2 \log \log n}} \right\} \rightarrow [-1, 1] \right) = 1 \quad (3.2)$$

when  $\mathbb{V}$  is continuous.

The proof of Theorem 3.1 is based on the the following self-normalized moderated deviation.

**Theorem 3.2** *Suppose conditions (I)-(III) in Theorem 3.1 are satisfied and that*

(IV)  $x_n \rightarrow \infty$  and  $x_n = o(\sqrt{n})$  as  $n \rightarrow \infty$ .

*Then*

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(S_n \geq x_n V_n) = -\frac{1}{2}. \quad (3.3)$$

*Further, it also holds that*

$$\limsup_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(S_n \geq x_n V_n) \leq -\frac{1}{2}$$

*if the condition  $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0$  is replaced by  $\widehat{\mathbb{E}}[X] \leq 0$ .*

**Remark 3.1** *If  $\widehat{\mathbb{E}}[X^2] < \infty$ ,  $\widehat{\mathcal{E}}[X^2] > 0$  and  $\widehat{\mathbb{E}}[(X^2 - c)^2] \rightarrow 0$  as  $c \rightarrow \infty$ , then conditions (I)-(III) are satisfied. Note*

$$\widehat{\mathbb{E}}[X^2 \wedge x^2] \leq \widehat{\mathbb{E}}[X^2 \wedge (kx)^2] \leq \widehat{\mathbb{E}}[X^2 \wedge x^2] + k^2 x^2 \mathbb{V}(|X| > x), \quad k \geq 1.$$

*The condition (I) implies that  $l(x)$  is slowly varying as  $x \rightarrow \infty$ , i.e., for all  $c > 0$ ,  $l(cx)/l(x) \rightarrow 1$  as  $x \rightarrow \infty$ . Further*

$$\frac{\widehat{\mathbb{E}}^*[X^2 I\{|X| \leq x\}]}{l(x)} \rightarrow 1, \quad \widehat{\mathbb{E}}[|X|^r \wedge x^r] = o(x^{r-2} l(x)), \quad r > 2,$$

$$C_{\mathbb{V}}(|X|^r I\{|X| \geq x\}) = \int_{x^r}^{\infty} \mathbb{V}(|X|^r \geq y) dy = o(x^{2-r} l(x)), \quad 0 < r < 2.$$

*When Conditions (I) and (III) are satisfied,*

$$\widehat{\mathbb{E}}[(|X| - x)^+] \leq \widehat{\mathbb{E}}^*[|X| I\{|X| \geq x\}] \leq C_{\mathbb{V}}(|X| I\{|X| \geq x\}) = o(x^{-1} l(x)).$$

## 4 Proofs

Let

$$b = \inf\{x \geq 0 : l(x) > 0\},$$

$$z_n = \inf \left\{ s : s \geq b + 1, \frac{l(s)}{s^2} \leq \frac{x_n^2}{n} \right\}.$$



Then  $z_n \rightarrow \infty$ ,  $nl(z_n) = x_n^2 z_n^2$ . We follow the main idea of Shao (1997,1999) and Jing, Shao and Wang (2003). The main difference is that  $-\widehat{\mathbb{E}}[\cdot]$  and  $\widehat{\mathbb{E}}[-\cdot]$  may be different and the conjugate method is not available. The proof of Theorem 3.2 will be completed via four propositions.

**Proposition 4.1** *We have*

$$\mathbb{V}\left(S_n \geq x_n V_n, V_n^2 \geq 9nl(z_n)\right) \leq \exp\left\{-x_n^2 + o(x_n)\right\}. \quad (4.1)$$

**Proof.** Observe that  $\widehat{\mathbb{E}}[X_1^2 \wedge z_n^2] = l(z_n)$ , and

$$\begin{aligned} & \mathbb{V}\left(S_n \geq x_n V_n, V_n^2 \geq 9nl(z_n)\right) \\ & \leq \mathbb{V}\left(\sum_{i=1}^n ((-z_n) \vee X_i \wedge z_n) \geq x_n V_n/2, V_n^2 \geq 9nl(z_n)\right) \\ & \quad + \mathbb{V}\left(\sum_{i=1}^n X_i I\{|X_i| > z_n\} \geq x_n V_n/2\right) \\ & \leq \mathbb{V}\left(\sum_{i=1}^n ((-z_n) \vee X_i \wedge z_n) \geq \frac{3}{2}x_n (nl(z_n))^{1/2}\right) \\ & \quad + \mathbb{V}\left(\sum_{i=1}^n I\{|X_i| > z_n\} \geq \frac{1}{4}x_n\right) \\ & =: J_1 + J_2. \end{aligned}$$

Let  $b_n = 1/z_n$ . As for  $J_1$ , by noting

$$e^x \leq 1 + x + \frac{x^2}{2} + \frac{|x|^3}{6}e^{|x|},$$

we have

$$\begin{aligned}
J_1 &\leq \exp \left\{ -\frac{3}{2} \frac{x_n (nl(z_n))^{1/2}}{z_n} \right\} \widehat{\mathbb{E}} \exp \left\{ \frac{1}{z_n} \sum_{i=1}^n ((-z_n) \vee X_i \wedge z_n) \right\} \\
&\leq \exp \left\{ -\frac{3}{2} x_n^2 \right\} \left[ \widehat{\mathbb{E}} \exp \left\{ \frac{1}{z_n} ((-z_n) \vee X_i \wedge z_n) \right\} \right]^n \\
&\leq \exp \left\{ -\frac{3}{2} x_n^2 \right\} \left[ 1 + \frac{\widehat{\mathbb{E}} [((-z_n) \vee X_i \wedge z_n)]}{z_n} + \frac{1}{2} \frac{\widehat{\mathbb{E}} [X_1^2 \wedge z_n^2]}{z_n^2} + \frac{e}{6} \frac{\widehat{\mathbb{E}} [|X_1|^3 \wedge z_n^3]}{z_n^3} \right]^n \\
&\leq \exp \left\{ -\frac{3}{2} x_n^2 \right\} \left[ 1 + \frac{\widehat{\mathbb{E}} [X_1] + \widehat{\mathbb{E}} [(|X_1| - z_n)^+]}{z_n} + \frac{1}{2} \frac{l(z_n)}{z_n^2} + \frac{e}{6} \frac{o(z_n l(z_n))}{z_n^3} \right]^n \\
&\leq \exp \left\{ -\frac{3}{2} x_n^2 \right\} \left[ 1 + \frac{1}{2} \frac{l(z_n)}{z_n^2} + \frac{o(l(z_n))}{z_n^2} \right]^n \\
&\leq \exp \left\{ -\frac{3}{2} x_n^2 \right\} \exp \left\{ \frac{1}{2} \frac{nl(z_n)}{z_n^2} + \frac{o(nl(z_n))}{z_n^2} \right\} \leq \exp \{-x_n^2 + o(x_n^2)\}.
\end{aligned}$$

As for  $J_2$ , we have

$$\begin{aligned}
J_2 &\leq \exp \left\{ -\frac{t}{4} x_n^2 \right\} \widehat{\mathbb{E}}^* \exp \left\{ t \sum_{i=1}^n I\{|X_i| > z_n\} \right\} \\
&\leq \exp \left\{ -\frac{t}{4} x_n^2 \right\} \widehat{\mathbb{E}}^* \left[ \prod_{i=1}^n (1 + (e^t - 1) I\{|X_i| > z_n\}) \right] \\
&\leq \exp \left\{ -\frac{t}{4} x_n^2 \right\} \widehat{\mathbb{E}} \left[ \prod_{i=1}^n (1 + (e^t - 1) h(|X_i|/z_n)) \right] \\
&= \exp \left\{ -\frac{t}{4} x_n^2 \right\} \prod_{i=1}^n \left( 1 + (e^t - 1) \widehat{\mathbb{E}} [h(|X_i|/z_n)] \right) \\
&\leq \exp \left\{ -\frac{t}{4} x_n^2 \right\} \prod_{i=1}^n (1 + (e^t - 1) \mathbb{V}(|X_i| > z_n/2)) \\
&\leq \exp \left\{ -\frac{t}{4} x_n^2 + (e^t - 1) n \mathbb{V}(|X_1| > z_n/2) \right\} \\
&\leq \exp \left\{ -\frac{t}{4} x_n^2 + (e^t - 1) n \frac{o(l(z_n))}{z_n^2} \right\} \leq \exp \{-2x_n^2 + o(x_n^2)\} \tag{4.2}
\end{aligned}$$

for  $t$  being chosen large enough, where  $h(\cdot)$  is a Lipschitz function such that  $I\{x > 1\} \leq h(x) \leq I\{x > 1/2\}$ . The proof is completed.  $\square$

Let  $\lambda$  and  $\theta > 0$  be two real numbers. Define

$$f(s) = e^{\lambda s - \theta s^2}. \tag{4.3}$$

Then

$$\begin{aligned}
f'(s) &= f(s)(\lambda - 2\theta s), \quad f'(0) = \lambda \\
f''(s) &= f(s)((\lambda - 2\theta s)^2 - 2\theta), \quad f''(0) = \lambda^2 - 2\theta, \\
f'''(s) &= f(s)(\lambda - 2\theta s)((\lambda - 2\theta s)^2 - 6\theta), \\
f^{(4)}(s) &= f(s)((\lambda - 2\theta s)^4 - 12\theta(\lambda - 2\theta s)^2 + 12\theta^2).
\end{aligned}$$

It is easily verified that

$$|f(s)| \leq e^{\frac{\lambda^2}{4\theta}}, \quad |f''(s)| \leq 4e^{-\frac{3}{2}}\theta e^{\frac{\lambda^2}{4\theta}}, \quad |f'''(s)| \leq 3\theta^{3/2}e^{\frac{\lambda^2}{4\theta}}.$$

It follows that

$$\begin{aligned}
f(s) &= 1 + \lambda s + \left(\frac{\lambda^2}{2} - \theta\right)(s^2 \wedge 1) + g(s) \quad \text{with} \\
|g(s)| &\leq \frac{1}{2}\theta^{3/2}e^{\frac{\lambda^2}{4\theta}}(|s| \wedge 1)^3 + |\lambda|(|s| - 1)^+ + e^{\frac{\lambda^2}{4\theta}}I\{|s| > 1\}.
\end{aligned}$$

For non-negative  $b$  and a random variable  $\xi$ , we have

$$\begin{aligned}
\widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} &= \widehat{\mathbb{E}} \left[ 1 + \lambda b\xi + \left(\frac{\lambda^2}{2} - \theta\right)((b\xi)^2 \wedge 1) + g(b\xi) \right] \\
&\leq 1 + \widehat{\mathbb{E}}[\lambda b\xi] + \widehat{\mathbb{E}} \left[ \left(\frac{\lambda^2}{2} - \theta\right)((b\xi)^2 \wedge 1) + g(b\xi) \right] \\
&\leq 1 + \widehat{\mathbb{E}} \left[ \left(\frac{\lambda^2}{2} - \theta\right)((b\xi)^2 \wedge 1) + g(b\xi) \right] \\
&\leq 1 + \left(\frac{\lambda^2}{2} - \theta\right)^+ \widehat{\mathbb{E}} [(b\xi)^2 \wedge 1] - \left(\frac{\lambda^2}{2} - \theta\right)^- \widehat{\mathcal{E}} [(b\xi)^2 \wedge 1] + \widehat{\mathbb{E}}[|g(b\xi)|]
\end{aligned}$$

if  $\widehat{\mathbb{E}}[\xi] \leq 0$ , and

$$\begin{aligned}
&\widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} \\
&\geq 1 + \widehat{\mathcal{E}}[\lambda b\xi] + \widehat{\mathbb{E}} \left[ \left(\frac{\lambda^2}{2} - \theta\right)((b\xi)^2 \wedge 1) + g(b\xi) \right] \\
&\geq 1 + \left(\frac{\lambda^2}{2} - \theta\right)^+ \widehat{\mathbb{E}} [(b\xi)^2 \wedge 1] - \left(\frac{\lambda^2}{2} - \theta\right)^- \widehat{\mathcal{E}} [(b\xi)^2 \wedge 1] - \widehat{\mathbb{E}}[|g(b\xi)|]
\end{aligned}$$

if  $\widehat{\mathcal{E}}[\xi] \geq 0$ . Hence we obtain the following lemma.

**Lemma 4.1** *Suppose that  $b$  is a positive number and  $\xi$  is a random variable. Then*

$$\begin{aligned}
&\widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} \\
&\leq 1 + \left(\frac{\lambda^2}{2} - \theta\right)^+ \widehat{\mathbb{E}} [(b\xi)^2 \wedge 1] - \left(\frac{\lambda^2}{2} - \theta\right)^- \widehat{\mathcal{E}} [(b\xi)^2 \wedge 1] \\
&\quad + O_{\lambda, \theta} \left( \widehat{\mathbb{E}}[|b\xi|^3 \wedge 1] + \widehat{\mathbb{E}}[|b\xi| - 1]^+ + \mathbb{V}(|b\xi| > 1) \right)
\end{aligned}$$

if  $\widehat{\mathbb{E}}[\xi] \leq 0$ , and

$$\begin{aligned} & \widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} \\ & \geq 1 + \left( \frac{\lambda^2}{2} - \theta \right)^+ \widehat{\mathbb{E}} [((b\xi)^2 \wedge 1)] - \left( \frac{\lambda^2}{2} - \theta \right)^- \widehat{\mathcal{E}} [((b\xi)^2 \wedge 1)] \\ & \quad + O_{\lambda, \theta} \left( \widehat{\mathbb{E}}[|b\xi|^3 \wedge 1] + \widehat{\mathbb{E}}[ (|b\xi| - 1)^+ ] + \mathbb{V}(|b\xi| > 1) \right) \end{aligned}$$

if  $\widehat{\mathcal{E}}[\xi] \geq 0$ , where  $|O_{\lambda, \theta}| \leq \frac{1}{2}\theta^{3/2}e^{\frac{\lambda^2}{4\theta}} + |\lambda| + e^{\frac{\lambda^2}{4\theta}}$ .

Choose  $b = b_n = z_n^{-1}$  if  $\frac{\lambda^2}{2} - \theta \geq 0$  and  $b = \underline{b}_n = \frac{1}{z_n} \frac{\widehat{\mathbb{E}}[X_1^2 \wedge z_n^2]}{\widehat{\mathcal{E}}[X_1^2 \wedge z_n^2]}$  if  $\frac{\lambda^2}{2} - \theta < 0$ . Then  $z_n^{-2} \leq \underline{b}_n \leq Cz_n^{-1}$ , and so

$$\begin{aligned} & \widehat{\mathbb{E}} \exp \{ \lambda(bX_1) - \theta(bX_1)^2 \} \\ & = 1 + \left( \frac{\lambda^2}{2} - \theta \right)^+ \underline{b}_n^2 \widehat{\mathbb{E}} [X_1^2 \wedge \underline{b}_n^{-2}] - \left( \frac{\lambda^2}{2} - \theta \right)^- \underline{b}_n^2 \widehat{\mathcal{E}} [X_1^2 \wedge \underline{b}_n^{-2}] \\ & \quad + O_{\lambda, \theta} \left( \widehat{\mathbb{E}}[|bX_1|^3 \wedge 1] + \widehat{\mathbb{E}}[ (|bX_1| - 1)^+ ] + \mathbb{V}(|bX_1| > 1) \right) \\ & = 1 + \left( \frac{\lambda^2}{2} - \theta \right) \frac{l(z_n)}{z_n^2} + \frac{o(l(z_n))}{z_n^2} O_{\lambda, \theta} \\ & = 1 + \left( \frac{\lambda^2}{2} - \theta \right) \frac{x_n^2}{n} + o\left(\frac{x_n^2}{n}\right) O_{\lambda, \theta} = \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) \frac{x_n^2}{n} + o\left(\frac{x_n^2}{n}\right) O_{\lambda, \theta} \right\}. \end{aligned}$$

Hence we obtain the following lemma.

**Lemma 4.2** Let  $b = b_n = \frac{1}{z_n}$  if  $\frac{\lambda^2}{2} - \theta \geq 0$  and  $b = \underline{b}_n = \frac{1}{z_n} \frac{\widehat{\mathbb{E}}[X_1^2 \wedge z_n^2]}{\widehat{\mathcal{E}}[X_1^2 \wedge z_n^2]}$  if  $\frac{\lambda^2}{2} - \theta < 0$ .

Then

$$\widehat{\mathbb{E}} \exp \{ \lambda(bS_n) - \theta(bV_n)^2 \} = \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) x_n^2 + o(x_n^2) O_{\lambda, \theta} \right\}.$$

Further, it also holds that

$$\widehat{\mathbb{E}} \exp \{ \lambda(bS_n) - \theta(bV_n)^2 \} \leq \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) x_n^2 + o(x_n^2) O_{\lambda, \theta} \right\}$$

if the condition  $\widehat{\mathbb{E}}[X] = \widehat{\mathcal{E}}[X] = 0$  is replaced by  $\widehat{\mathbb{E}}[X] \leq 0$ .

**Proposition 4.2** For  $0 < \delta < 1$ ,

$$\mathbb{V}(S_n \geq x_n V_n, \delta n l(z_n) \leq V_n^2 \leq 9 n l(z_n)) \leq \exp \{ -x_n^2 + o(x_n^2) \}.$$

**Proof.** Let  $1 < \theta < 2$ ,  $b = b_n = z_n^{-1}$ . Then by Lemma 4.2,

$$\begin{aligned}
& \mathbb{V}(S_n \geq x_n V_n, \delta n l(z_n) \leq V_n^2 \leq 9 n l(z_n)) \\
& \leq \mathbb{V}\left(2 \frac{x_n}{V_n} S_n - \left(\frac{x_n}{V_n} V_n\right)^2 \geq x_n^2, \frac{1}{3} b_n \leq \frac{x_n}{V_n} \leq \delta^{-1/2} b_n\right) \\
& \leq \mathbb{V}\left(\sup_{3^{-1} \leq a \leq \delta^{-1/2}} (2 a b_n S_n - (a b_n V_n)^2) \geq x_n^2\right) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \mathbb{V}\left(\sup_{\theta^j \leq a \leq \theta^{j+1}} (2 a b_n S_n - (a b_n V_n)^2) \geq x_n^2\right) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \mathbb{V}(2 \theta^{j+1} b_n S_n - (\theta^j b_n V_n)^2 \geq x_n^2) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \exp\left\{-\frac{x_n^2}{2}\right\} \widehat{\mathbb{E}} \exp\left\{\theta^{j+1} b_n S_n - \frac{1}{2} \theta^{2j} (b_n V_n)^2\right\} \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \exp\left\{-\frac{x_n^2}{2}\right\} \exp\left\{\left(\frac{1}{2} \theta^{2j+2} - \frac{1}{2} \theta^{2j}\right) x_n^2 + o(x_n^2)\right\} \\
& \leq \left(\frac{\log \delta^{-1/2} + \log 3}{\log \theta} + 1\right) \exp\left\{-\frac{x_n^2}{2} + \frac{1}{2} (\theta^2 - 1) \delta^{-1} x_n^2 + o(x_n^2)\right\}.
\end{aligned}$$

Let  $\theta^2 = 1 + x_n^{-1}$ . Then

$$\begin{aligned}
& \mathbb{V}(S_n \geq x_n V_n, \delta n l(z_n) \leq V_n^2 \leq 9 n l(z_n)) \\
& \leq (\log \delta^{-1/2} + 2) 2 x_n \exp\left\{-\frac{x_n^2}{2} + \frac{1}{2} \delta^{-1} x_n + o(x_n^2)\right\} \\
& \leq \exp\left\{-\frac{x_n^2}{2} + o(x_n^2)\right\}. \quad \square
\end{aligned}$$

Next, we consider the event  $\{S_n \geq x_n V_n, V_n^2 \leq \delta n l(z_n)\}$ . We need Bernstein's type inequalities.

**Lemma 4.3** Suppose that  $\{Y_n\}$  is a sequence of random variables on  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$  with  $|Y_n| \leq a$ . Set  $T_n = Y_1 + \cdots + Y_n$ ,  $B_n^2 = \sum_{i=1}^n \widehat{\mathbb{E}}[(Y_i - \widehat{\mathbb{E}}[Y_i])^2]$ . Then

$$\begin{aligned}
& \mathbb{V}\left(\sum_{i=1}^n (Y_i - \widehat{\mathbb{E}}[Y_i]) \geq x\right) \\
& \leq \exp\left\{-\frac{x^2}{2(B_n^2 + 2ax)}\right\} \leq \exp\left\{-\frac{x^2}{8 \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i^2] + 4ax}\right\}, \tag{4.4}
\end{aligned}$$

$$\begin{aligned}
& \mathbb{V} \left( \max_{i \leq n} T_i \geq x + \max_{k \leq n} \sum_{i=k+1}^n \widehat{\mathbb{E}}[-Y_i] + \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i] \right) \\
& \leq 4 \exp \left\{ -\frac{x^2}{2(B_n^2 + 2ax)} \right\} \leq 4 \exp \left\{ -\frac{x^2}{8 \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i^2] + 4ax} \right\}
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\mathcal{V} \left( \sum_{i=1}^n (Y_i - \widehat{\mathcal{E}}[Y_i]) \geq x \right) & \leq \exp \left\{ -\frac{x^2}{2(\sum_{i=1}^n \widehat{\mathbb{E}}[(Y_i - \widehat{\mathcal{E}}[Y_i])^2] + 2ax)} \right\} \\
& \leq \exp \left\{ -\frac{x^2}{8 \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i^2] + 4ax} \right\}.
\end{aligned} \tag{4.6}$$

**Proof.** Suppose  $0 < \lambda \cdot 2a \leq c < 1$ . Then

$$\begin{aligned}
\widehat{\mathbb{E}} \left[ e^{\lambda(Y_i - \widehat{\mathbb{E}}[Y_i])} \right] & \leq \widehat{\mathbb{E}} \left[ 1 + \lambda(Y_i - \widehat{\mathbb{E}}[Y_i]) + \frac{\lambda^2}{2}(Y_i - \widehat{\mathbb{E}}[Y_i])^2 \left( 1 + \sum_{k=3}^{\infty} (2\lambda a)^{k-2} \right) \right] \\
& \leq 1 + \lambda \widehat{\mathbb{E}}[(Y_i - \widehat{\mathbb{E}}[Y_i])] + \frac{\lambda^2 \widehat{\mathbb{E}}[(Y_i - \widehat{\mathbb{E}}[Y_i])^2]}{2(1-c)} \\
& \leq \exp \left\{ \frac{\lambda^2 \widehat{\mathbb{E}}[(Y_i - \widehat{\mathbb{E}}[Y_i])^2]}{2(1-c)} \right\}.
\end{aligned}$$

So,

$$\widehat{\mathbb{E}} \left[ e^{\lambda \sum_{i=1}^n (Y_i - \widehat{\mathbb{E}}[Y_i])} \right] = \prod_{i=1}^n \widehat{\mathbb{E}} \left[ e^{\lambda(Y_i - \widehat{\mathbb{E}}[Y_i])} \right] \leq \exp \left\{ \frac{\lambda^2 B_n^2}{2(1-c)} \right\}.$$

Letting  $\lambda = \frac{x}{B_n^2 + 2ax}$  and  $c = \frac{2ax}{B_n^2 + 2ax}$  yields

$$\begin{aligned}
\mathbb{V} \left( \sum_{i=1}^n (Y_i - \widehat{\mathbb{E}}[Y_i]) \geq x \right) & \leq e^{-\lambda x} \widehat{\mathbb{E}} \left[ e^{\lambda \sum_{i=1}^n (Y_i - \widehat{\mathbb{E}}[Y_i])} \right] \\
& \leq e^{-\lambda x} \exp \left\{ \frac{\lambda^2 B_n^2}{2(1-c)} \right\} \leq \exp \left\{ -\frac{x^2}{2(B_n^2 + 2ax)} \right\}.
\end{aligned}$$

As for (4.5), we first show that

$$\widehat{\mathbb{E}} \exp \left\{ \lambda \max_{k \leq n} \sum_{i=1}^k (Y_i - \widehat{\mathcal{E}}[Y_i]) \right\} \leq 4 \widehat{\mathbb{E}} \exp \left\{ \lambda \sum_{i=1}^n (Y_i - \widehat{\mathcal{E}}[Y_i]) \right\}. \tag{4.7}$$

Let

$$Q_0 = 0, \quad Q_k = \exp \left\{ \frac{\lambda}{2} \sum_{i=1}^k (Y_i - \widehat{\mathcal{E}}[Y_i]) \right\}, \quad M_k = \max_{i \leq k} Q_i.$$

Then  $M_k \geq Q_k \geq 0$  and

$$Q_n M_n = \sum_{k=0}^{n-1} Q_{k+1} (M_{k+1} - M_k) + \sum_{k=0}^{n-1} (Q_{k+1} - Q_k) M_k.$$

Note that  $Q_{k+1} = M_{k+1}$  when  $M_{k+1} \neq M_k$ . So

$$Q_{k+1} (M_{k+1} - M_k) = M_{k+1} (M_{k+1} - M_k) \geq \frac{1}{2} M_{k+1}^2 - \frac{1}{2} M_k^2.$$

It follows that

$$\begin{aligned} Q_n M_n &\geq \frac{1}{2} M_n^2 + \sum_{k=0}^{n-1} Q_k M_k \left( e^{\frac{\lambda}{2} (Y_{k+1} - \widehat{\mathcal{E}}[Y_{k+1}])} - 1 \right) \\ &\geq \frac{1}{2} M_n^2 + \frac{\lambda}{2} \sum_{k=0}^{n-1} Q_k M_k (Y_{k+1} - \widehat{\mathcal{E}}[Y_{k+1}]). \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} \widehat{\mathbb{E}}[M_n^2] &\leq \widehat{\mathbb{E}}[Q_n M_n] + \frac{\lambda}{2} \sum_{k=0}^{n-1} \widehat{\mathbb{E}}[Q_k M_k] \widehat{\mathbb{E}}[\widehat{\mathcal{E}}[Y_{k+1}] - Y_{k+1}] \\ &= \widehat{\mathbb{E}}[Q_n M_n] \leq \widehat{\mathbb{E}}\left[\frac{1}{4} M_n^2 + Q_n^2\right] \leq \frac{1}{4} \widehat{\mathbb{E}}[M_n^2] + \widehat{\mathbb{E}}[Q_n^2]. \end{aligned}$$

Hence  $\widehat{\mathbb{E}}[M_n^2] \leq 4\widehat{\mathbb{E}}[Q_n^2]$ . The proof of (4.7) is now completed.

Now, by (4.7),

$$\begin{aligned} \widehat{\mathbb{E}} \exp \left\{ \lambda \max_{k \leq n} T_k \right\} &\leq \exp \left\{ \lambda \max_{k \leq n} \sum_{i=1}^k \widehat{\mathcal{E}}[Y_i] \right\} \widehat{\mathbb{E}} \exp \left\{ \lambda \max_{k \leq n} \sum_{i=1}^k (Y_i - \widehat{\mathcal{E}}[Y_i]) \right\} \\ &\leq 4 \exp \left\{ \lambda \max_{k \leq n} \sum_{i=1}^k \widehat{\mathcal{E}}[Y_i] \right\} \widehat{\mathbb{E}} \exp \left\{ \lambda \sum_{i=1}^n (Y_i - \widehat{\mathcal{E}}[Y_i]) \right\} \\ &= 4 \exp \left\{ \lambda \max_{k \leq n} \sum_{i=1}^k \widehat{\mathcal{E}}[Y_i] - \sum_{i=1}^n \widehat{\mathcal{E}}[Y_i] + \lambda \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i] \right\} \widehat{\mathbb{E}} \exp \left\{ \lambda \sum_{i=1}^n (Y_i - \widehat{\mathbb{E}}[Y_i]) \right\} \\ &\leq 4 \exp \left\{ \lambda \max_{k \leq n} \sum_{i=k+1}^n \widehat{\mathbb{E}}[-Y_i] + \lambda \sum_{i=1}^n \widehat{\mathbb{E}}[Y_i] \right\} \exp \left\{ \frac{\lambda^2 B_n^2}{2(1-c)} \right\}, \end{aligned}$$

when  $2a\lambda \leq c < 1$ . The remainder proof is the same as that of (4.4).

The proof of (4.6) is similar to that of (4.4), if the following facts are noted:

$$\mathcal{V} \left( \sum_{i=1}^n (Y_i - \widehat{\mathcal{E}}[Y_i]) \geq x \right) \leq e^{-\lambda x} \widehat{\mathcal{E}} \left[ e^{\lambda \sum_{i=1}^n (Y_i - \widehat{\mathcal{E}}[Y_i])} \right] = e^{-\lambda x} \prod_{i=1}^n \widehat{\mathcal{E}} \left[ e^{\lambda (Y_i - \widehat{\mathcal{E}}[Y_i])} \right]$$

and

$$\begin{aligned}\widehat{\mathcal{E}}\left[e^{\lambda(Y_i - \widehat{\mathbb{E}}[Y_i])}\right] &\leq \widehat{\mathcal{E}}\left[1 + \lambda(Y_i - \widehat{\mathbb{E}}[Y_i]) + \frac{\lambda^2(Y_i - \widehat{\mathbb{E}}[Y_i])^2}{2(1-c)}\right] \\ &\leq 1 + \lambda\widehat{\mathcal{E}}[(Y_i - \widehat{\mathcal{E}}[Y_i])] + \frac{\lambda^2\widehat{\mathbb{E}}[(Y_i - \widehat{\mathbb{E}}[Y_i])^2]}{2(1-c)}, \quad 0 < 2a\lambda < 1,\end{aligned}$$

where the last inequality is due the fact that  $\widehat{\mathcal{E}}[X + Y] \leq \widehat{\mathcal{E}}[X] + \widehat{\mathbb{E}}[Y]$ .  $\square$

**Proposition 4.3** Suppose  $0 < \delta < r^{-2}$ . For  $n$  large enough,

$$\mathbb{V}(V_n^2 \leq \delta nl(z_n)) \leq \exp\{-2x_n^2\}.$$

**Proof.** Let  $\varepsilon = r^{-2} - \delta$ . Applying the Bernstein inequality (4.4) again yields

$$\begin{aligned}\mathbb{V}(V_n^2 \leq \delta nl(z_n)) &\leq \mathbb{V}\left(\sum_{i=1}^n X_i^2 \wedge z_n^2 \leq \delta nl(z_n)\right) \\ &\leq \mathbb{V}\left(\sum_{i=1}^n (-X_i^2 \wedge z_n^2 + \widehat{\mathcal{E}}[X_i^2 \wedge z_n^2]) \geq n\widehat{\mathcal{E}}[X_1^2 \wedge z_n^2] - \delta nl(z_n)\right) \\ &\leq \mathbb{V}\left(\sum_{i=1}^n (-X_i^2 \wedge z_n^2 + \widehat{\mathcal{E}}[X_i^2 \wedge z_n^2]) \geq r^{-2}n\widehat{\mathbb{E}}[X_1^2 \wedge z_n^2] - \delta nl(z_n)\right) \\ &= \mathbb{V}\left(\sum_{i=1}^n (-X_i^2 \wedge z_n^2 + \widehat{\mathcal{E}}[X_i^2 \wedge z_n^2]) \geq \varepsilon nl(z_n)\right) \\ &\leq \exp\left\{-\frac{(\varepsilon nl(z_n))^2}{8n\widehat{\mathbb{E}}[X_1^4 \wedge z_n^4] + 4z_n \cdot \varepsilon nl(z_n)}\right\} \\ &\leq \exp\left\{-\frac{(\varepsilon nl(z_n))^2}{8n \cdot o(z_n^2 l(z_n)) + 4z_n \cdot \varepsilon nl(z_n)}\right\} \\ &\leq \exp\left\{-\frac{\varepsilon nl(z_n)}{o(z_n^2) + 4z_n}\right\} = \exp\left\{-\frac{x_n^2}{o(1)}\right\} \leq \exp\{-2x_n^2\}. \quad \square\end{aligned}\tag{4.8}$$

Now, the upper bound of (3.3) follows from Propositions 4.1-4.3 immediately. As for the lower bound, we let  $\underline{b}_n = \frac{1}{z_n}$  and  $\eta_n = 2\underline{b}_n S_n - (\underline{b}_n V_n)^2$ . Note

$$x_n V_n = \inf_b \frac{x_n^2 + b^2 V_n^2}{2b} \leq \frac{x_n^2 + \underline{b}_n^2 V_n^2}{2\underline{b}_n}.$$

We have

$$\mathbb{V}(S_n \geq x_n V_n) \geq \mathbb{V}\left(S_n \geq \frac{x_n^2 + \underline{b}_n^2 V_n^2}{2\underline{b}_n}\right) \geq \mathbb{V}(2\underline{b}_n S_n - (\underline{b}_n V_n)^2 \geq x_n^2).$$

The lower bound of (3.3) follows from the following proposition.



**Proposition 4.4** For any  $0 < \beta < 1$ ,

$$\liminf_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(\eta_n \geq (1 + \beta)x_n^2) \geq -\frac{(1 + \beta/4)^2}{2}. \quad (4.9)$$

**Proof.** By Lemma 4.2,

$$\widehat{\mathbb{E}}[e^{\lambda \eta_n}] = \exp \left\{ (2\lambda^2 - \lambda)x_n^2 + o(x_n^2) \right\}, \quad \lambda > 1/2.$$

That is,

$$\varphi(\lambda) = \lim_{n \rightarrow \infty} x_n^{-2} \ln \widehat{\mathbb{E}}[e^{\lambda \eta_n}] = 2\lambda^2 - \lambda, \quad \lambda > 1/2.$$

On the other hand, note that  $\mathbb{V}$  is sub-additive and  $\widehat{\mathbb{E}}[(e^{\lambda \eta_n} - c)^+] \leq c^{-1} \widehat{\mathbb{E}}[e^{2\lambda \eta_n}] \rightarrow 0$  as  $c \rightarrow \infty$ . We have

$$\begin{aligned} \widehat{\mathbb{E}}[e^{\lambda \eta_n}] &\leq \int_0^\infty \mathbb{V}(e^{\lambda \eta_n} > t) dt \\ &\leq 1 + \int_1^\infty \frac{\widehat{\mathbb{E}}[e^{(1+\epsilon)\lambda \eta_n}]}{t^{1+\epsilon}} dt \leq 1 + \frac{1}{\epsilon} \widehat{\mathbb{E}}[e^{(1+\epsilon)\lambda \eta_n}]. \end{aligned} \quad (4.10)$$

It follows that

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \int_0^\infty \mathbb{V}(e^{\lambda \eta_n} > t) dt = 2\lambda^2 - \lambda, \quad \lambda > 1/2.$$

Let  $G_n(t) = 1 - \mathbb{V}(\eta_n > t)$  and  $F_n(t) = \lim_{y \searrow t} G_n(t)$ . It is easy to verify that  $F_n(t)$  is a probability distribution function. Let  $\xi_n$  be a random variable on the probability space  $(\Omega, \mathcal{F}, P)$  with the distribution  $F_n(t)$ . Then

$$\begin{aligned} \int_0^\infty \mathbb{V}(e^{\lambda \eta_n} > t) dt &= \int_{-\infty}^\infty \lambda e^{\lambda t} \mathbb{V}(\eta_n > t) dt \\ &= \int_{-\infty}^\infty \lambda e^{\lambda t} P(\xi_n > t) dt = E_P[e^{\lambda \xi_n}]. \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln E_P[e^{\lambda \xi_n}] = \varphi(\lambda) = 2\lambda^2 - \lambda, \quad \lambda > 1/2.$$

Note for  $x \in \{y : \varphi'(\lambda) = y, \exists \lambda > 1/2\} = (1, \infty)$ ,

$$I(x) = \sup_{\lambda > 1/2} \{\lambda x - \varphi(\lambda)\} = \frac{(x+1)^2}{8}.$$

By the Gärtner-Ellis Theorem (cf. Dembo and Zeitouni (1998)),

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(\eta_n \geq (1 + \beta)x_n^2) \\
& \geq \liminf_{n \rightarrow \infty} x_n^{-2} \ln \{1 - F_n(1 + \beta/2)x_n^2\} \\
& \geq \liminf_{n \rightarrow \infty} x_n^{-2} \ln P(\xi_n/x_n^2 > 1 + \beta/2) \\
& \geq - \inf_{x \in (1+\beta/2, \infty)} I(x) = -\frac{(1 + \beta/4)^2}{2}. \quad \square
\end{aligned}$$

The proof of Theorem 3.3 is now completed.

Now, we begin the proof of the self-normalized law of the iterated logarithm.

**Proof of Theorem 3.1.** We first show (3.1). That is

$$\limsup_{n \rightarrow \infty} \frac{|S_n|}{V_n \sqrt{2 \log \log n}} \leq 1 \quad a.s. \quad \mathbb{V}. \quad (4.11)$$

Let  $m_k = \lceil e^{k/(\log \log k)^2} \rceil$ ,  $x_k = (2 \log \log m_k)^{1/2}$ . Then  $x_k \sim (2 \log k)^{1/2}$ . Observe that for  $0 < \epsilon < 1/2$ ,

$$\begin{aligned}
& \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} \frac{S_n}{V_n} \geq (1 + 7\epsilon)x_k \right) \\
& \leq \mathbb{V} \left( \frac{S_{m_k}}{V_{m_k}} \geq (1 + 2\epsilon)x_k \right) + \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} \frac{S_n - S_{m_k}}{V_{m_k}} \geq 5\epsilon x_k \right). \quad (4.12)
\end{aligned}$$

By Theorem 3.3,

$$\mathbb{V} \left( \frac{S_{m_k}}{V_{m_k}} \geq (1 + 2\epsilon)x_k \right) \leq \exp \{ -(1 + 2\epsilon)x_k^2/2 \} \leq k^{-1-\epsilon}$$

for every sufficiently large  $k$ . We estimate the second term in the right-hand side of (4.12) below. Let  $\bar{z}_k$  be the number such that

$$\frac{l(\bar{z}_k)}{\bar{z}_k^2} = \frac{x_k^2}{m_{k+1} - m_k},$$

and denote  $T_n = \sum_{i=m_k+1}^n (-\bar{z}_k^2) \vee X_i \wedge \bar{z}_k^2$ . Then

$$\frac{\bar{z}_k x_k \sqrt{m_{k+1} l(\bar{z}_k)}}{x_k^2 \bar{z}_k^2} = \frac{\sqrt{m_{k+1}}}{\sqrt{m_{k+1} - m_k}} \rightarrow \infty.$$

Let  $0 < \delta^2 < r^{-2}/4$ . Observe that

$$\begin{aligned}
& \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} \frac{S_n - S_{m_k}}{V_{m_k}} \geq 5\epsilon x_k \right) \\
& \leq \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} T_n \geq 2\epsilon x_k V_{m_k}, V_{m,k}^2 > \delta^2 m_{k+1} l(\bar{z}_k) \right) \\
& \quad + \mathbb{V} (V_{m_k}^2 \leq \delta^2 m_{k+1} l(\bar{z}_k)) + \mathbb{V} \left( \sum_{n=m_k+1}^{m_{k+1}} (|X_i| - \bar{z}_k)^+ \geq 3\epsilon x_k V_{m_k} \right) \\
& \leq \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} T_n \geq 2\epsilon \delta x_k \sqrt{m_{k+1} l(\bar{z}_k)} \right) \\
& \quad + \mathbb{V} (V_{m_k}^2 \leq \delta^2 m_{k+1} l(\bar{z}_k)) + \mathbb{V} \left( \sum_{n=m_k+1}^{m_{k+1}} I\{|X_i| > \bar{z}_k\} \geq (\epsilon x_k)^2 \right).
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{i=m_k+1}^{m_{k+1}} \widehat{\mathbb{E}} \left[ ((-\bar{z}_k) \vee X_i \wedge \bar{z}_k)^2 \right] = (m_{k+1} - m_k) l(\bar{z}_k) = x_k^2 \bar{z}_k^2, \\
& \sum_{i=m_k+1}^{m_{k+1}} \left| \widehat{\mathcal{E}} [(-\bar{z}_k) \vee X_i \wedge \bar{z}_k] \right| + \sum_{i=m_k+1}^{m_{k+1}} \left| \widehat{\mathbb{E}} [(-\bar{z}_k) \vee X_i \wedge \bar{z}_k] \right| \\
& \leq \sum_{i=m_k+1}^{m_{k+1}} \widehat{\mathbb{E}} [(|X_i| - \bar{z}_k)^+] = (m_{k+1} - m_k) \frac{o(l(\bar{z}_k))}{\bar{z}_k} \\
& = o(x_k^2 \bar{z}_k^2) = o \left( x_k \sqrt{m_{k+1} l(\bar{z}_k)} \right).
\end{aligned}$$

So, by the Bernstein inequality (4.5), for sufficiently large  $k$ ,

$$\begin{aligned}
& \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} T_n \geq 2\epsilon \delta x_k \sqrt{m_{k+1} l(\bar{z}_k)} \right) \\
& \leq 4 \exp \left\{ - \frac{(\epsilon \delta x_k \sqrt{m_{k+1} l(\bar{z}_k)})^2}{8x_k^2 \bar{z}_k^2 + 4\bar{z}_k \cdot \epsilon \delta x_k \sqrt{m_{k+1} l(\bar{z}_k)}} \right\} \\
& \leq 4 \exp \left\{ - \frac{(\epsilon \delta)^2 \frac{m_{k+1}}{m_{k+1}-m_k} x_k^4 \bar{z}_k^2}{8x_k^2 \bar{z}_k^2 + 4 \cdot \epsilon \delta x_k^2 \bar{z}_k^2 \sqrt{\frac{m_{k+1}}{m_{k+1}-m_k}}} \right\} \\
& \leq \exp \{-2x_k^2\} \leq k^{-2}.
\end{aligned}$$

Note  $\delta^2 < r^{-2}/4$ . Similar to (4.8), applying the Bernstein inequality (4.4) again yields

$$\begin{aligned}
& \mathbb{V} \left( V_{m_k}^2 \leq \delta^2 m_{k+1} l(\bar{z}_k) \right) \leq \mathbb{V} \left( \sum_{i=1}^{m_k} X_i^2 \wedge \underline{z}_k^2 \leq \delta^2 m_{k+1} l(\bar{z}_k) \right) \\
& \leq \mathbb{V} \left( \sum_{i=1}^{m_k} \left( -X_i^2 \wedge \underline{z}_k^2 + \widehat{\mathcal{E}}[X_i^2 \wedge \underline{z}_k^2] \right) \geq m_k \widehat{\mathcal{E}}[X_1^2 \wedge \underline{z}_k^2] - \delta^2 m_{k+1} l(\bar{z}_k) \right) \\
& \leq \mathbb{V} \left( \sum_{i=1}^{m_k} \left( -X_i^2 \wedge \underline{z}_k^2 + \widehat{\mathcal{E}}[X_i^2 \wedge \underline{z}_k^2] \right) \geq m_k r^{-2} \widehat{\mathbb{E}}[X_1^2 \wedge \underline{z}_k^2] - \delta^2 m_{k+1} l(\bar{z}_k) \right) \\
& \leq \mathbb{V} \left( \sum_{i=1}^{m_k} \left( -X_i^2 \wedge \underline{z}_k^2 + \widehat{\mathcal{E}}[X_i^2 \wedge \underline{z}_k^2] \right) \geq \delta^2 m_{k+1} l(\bar{z}_k) \right) \\
& \leq \exp \left\{ -\frac{(\delta^2 m_{k+1} l(\bar{z}_k))^2}{8 m_k \widehat{\mathbb{E}}[X_1^4 \wedge \bar{z}_k^4] + 4 \bar{z}_k \cdot \delta^2 m_{k+1} l(\bar{z}_k)} \right\} \\
& \leq \exp \left\{ -\frac{(\delta^2 m_{k+1} l(\bar{z}_k))^2}{8 m_k \cdot o(\bar{z}_k^2 l(\bar{z}_k)) + 4 \bar{z}_k \cdot \delta^2 m_{k+1} l(\bar{z}_k)} \right\} \\
& \leq \exp \left\{ -\frac{\delta^2 m_{k+1} l(\bar{z}_k)}{o(\bar{z}_k^2) + 4 \bar{z}_k} \right\} \leq \exp \{-2x_k^2\} \leq k^{-2}.
\end{aligned}$$

Finally, similar to (4.2) we have for sufficiently large  $t$ ,

$$\begin{aligned}
& \mathbb{V} \left( \sum_{n=m_k+1}^{m_{k+1}} I\{|X_i| > \bar{z}_k\} \geq (\epsilon x_k)^2 \right) \\
& \leq \exp \left\{ -t(\epsilon x_k)^2 + (e^t - 1) \frac{(m_{k+1} - m_k) o(l(\bar{z}_k))}{\bar{z}_k^2} \right\} \\
& \leq \exp \{-t(\epsilon x_k)^2 + o(x_k^2)\} \leq \exp\{-x_k^2\} \leq k^{-2}.
\end{aligned}$$

Combing the above inequalities yields

$$\sum_k \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} \frac{S_n}{V_n} \geq (1 + 7\epsilon) x_k \right) \leq \sum_k (k^{-1-\epsilon} + 3k^{-2}) < \infty.$$

Note the countable sub-additivity of  $\mathbb{V}$ . By the Borel-Cantelli lemma,

$$\begin{aligned}
& \mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \geq 1 + 8\epsilon \right) \\
& \leq \mathbb{V} \left( \max_{m_k \leq n \leq m_{k+1}} \frac{S_n}{V_n} \geq (1 + 7\epsilon) x_k, \text{ i.o.} \right) = 0.
\end{aligned}$$

Let  $\{\epsilon_i\}$  be a sequence with  $\epsilon_i \searrow 0$ . Then

$$\begin{aligned}
& \mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} > 1 \right) \\
& \leq \sum_{i=1}^{\infty} \mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \geq 1 + 8\epsilon_i \right) = 0
\end{aligned}$$

by countable sub-additivity of  $\mathbb{V}$ . (4.11) is proved.

As for (3.2), note

$$\frac{|X_n|}{V_n \sqrt{2n \log \log n}} \leq \frac{1}{\sqrt{2 \log \log n}} \rightarrow 0.$$

By (3.1) and Proposition 2.1 of Griffin and Kuelbs (1989), it is sufficient to show that

$$\mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \geq 1 \text{ and } \liminf_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \leq -1 \right) = 1. \quad (4.13)$$

Let  $n_k = [e^{k(\log k)^2}]$ ,  $x_n = \sqrt{2 \log \log n}$ . Observe that  $x_{n_k}^2 \sim 2 \log k$ . Next, we show that

$$\lim_{k \rightarrow \infty} \frac{V_{n_k}}{V_{n_{k+1}}} = 0 \text{ a.s. } \mathbb{V}. \quad (4.14)$$

By Proposition 4.3 we have for  $\delta < r^{-2}$ ,

$$\mathbb{V} (V_{n_k}^2 \leq \delta n_k l(z_{n_k})) \leq \exp\{-2x_{n_k}^2\} \leq k^{-2}.$$

By the Borel-Cantelli lemma, we have

$$\liminf_{k \rightarrow \infty} \frac{V_{n_k}^2}{n_k l(z_{n_k})} \geq \delta \text{ a.s. } \mathbb{V}. \quad (4.15)$$

Also, for sufficiently large  $k$ ,

$$\begin{aligned} & \mathbb{V} \left( \sum_{i=1}^{n_k} X_i^2 \wedge z_{n_k}^2 \geq \epsilon n_{k+1} l(z_{n_{k+1}}) \right) \\ & \leq \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 \wedge z_{n_k}^2 - \widehat{\mathbb{E}}[X_i^2 \wedge z_{n_k}^2]) \geq \epsilon n_{k+1} l(z_{n_{k+1}}) - n_k l(z_{n_k}) \right) \\ & \leq \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 \wedge z_{n_k}^2 - \widehat{\mathbb{E}}[X_i^2 \wedge z_{n_k}^2]) \geq \frac{\epsilon}{2} n_{k+1} l(z_{n_k}) \right) \\ & \leq \exp \left\{ -\frac{(\frac{\epsilon}{2} n_{k+1} l(z_{n_k}))^2}{8o(1)z_{n_k}^2 n_k l(z_{n_k}) + z_{n_k}^2 \cdot \frac{\epsilon}{2} n_{k+1} l(z_{n_k})} \right\} \leq \exp\{-x_{n_k}^2\} \leq k^{-2}. \end{aligned}$$

Finally, let  $a_k^2 = \frac{n_{k+1}}{n_k} z_{n_k}^2$

$$\begin{aligned} & \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 - z_{n_k}^2)^+ \geq \epsilon n_{k+1} l(z_{n_{k+1}}) \right) \\ & \leq \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 - z_{n_k}^2)^+ \geq \epsilon n_{k+1} l(z_{n_k}) \right) = \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 - z_{n_k}^2)^+ \geq \epsilon a_k^2 x_{n_k}^2 \right) \\ & \leq \mathbb{V} \left( \sum_{i=1}^{n_k} (X_i^2 - z_{n_k}^2)^+ \geq \epsilon a_k^2 x_{n_k}^2, \max_{i \leq n_k} |X_i| \leq a_k \right) + n_k \mathbb{V}(|X_1| \geq a_k) \\ & \leq \mathbb{V} \left( \sum_{i=1}^{n_k} I\{|X_i| > z_{n_k}\} \geq \epsilon x_{n_k}^2 \right) + n_k \mathbb{V}(|X_1| \geq a_k). \end{aligned}$$

Similar to similar to (4.2) we have

$$\mathbb{V} \left( \sum_{i=1}^{n_k} I\{|X_i| > z_{n_k}\} \geq \epsilon x_{n_k}^2 \right) \leq \exp\{-x_{n_k}^2\} \leq k^{-2}.$$

Observer that

$$\begin{aligned} n_k \mathbb{V}(|X_1| \geq a_k) &= o(1) n_k \frac{l(a_k)}{a_k^2} \leq c \frac{n_k^2 l(a_k)}{n_{k+1} z_{n_k}^2} \leq c x_{n_k}^2 \frac{n_k}{n_{k+1}} \frac{l(a_k)}{l(z_{n_k})} \\ &\leq c x_{n_k}^2 \left( \frac{n_k}{n_{k+1}} \right)^{1-\epsilon} \leq k^{-2}. \end{aligned}$$

Combing the above inequality and applying the Borel-Cantelli lemma yield

$$\lim_{k \rightarrow \infty} \frac{V_{n_k}^2}{n_{k+1} l(z_{n_{k+1}})} = 0 \text{ a.s. } \mathbb{V}. \quad (4.16)$$

Now, (4.14) follows from (4.15) and (4.16).

Hence, by (4.14) and (4.11) we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} &\geq \limsup_{k \rightarrow \infty} \frac{S_{n_k}}{V_{n_k} \sqrt{2 \log \log n_k}} \\ &\geq \limsup_{k \rightarrow \infty} \frac{S_{n_k} - S_{n_{k-1}}}{V_{n_k} \sqrt{2 \log \log n_k}} - \limsup_{k \rightarrow \infty} \frac{|S_{n_{k-1}}|}{V_{n_k} \sqrt{2 \log \log n_k}} \\ &= \limsup_{k \rightarrow \infty} \left( 1 - \frac{V_{n_{k-1}}^2}{V_{n_k}^2} \right)^{1/2} \frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \\ &\quad - \limsup_{k \rightarrow \infty} \frac{V_{n_{k-1}}}{V_{n_k}} \frac{|S_{n_{k-1}}|}{V_{n_{k-1}} \sqrt{2 \log \log n_k}} \end{aligned} \quad (4.17)$$

$$= \limsup_{k \rightarrow \infty} \frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \text{ a.s. } \mathbb{V} \quad (4.18)$$

and similarly,

$$\limsup_{n \rightarrow \infty} \frac{-S_n}{V_n \sqrt{2 \log \log n}} \geq \limsup_{k \rightarrow \infty} \frac{-(S_{n_k} - S_{n_{k-1}})}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \text{ a.s. } \mathbb{V}. \quad (4.19)$$

Let  $h(x)$  be a non-increasing Lipschitz function such that  $I\{x \geq 1 - 2\epsilon\} \geq h(x) \geq I\{x \geq 1 - \epsilon\}$ . Denote

$$\begin{aligned} \eta_{k,1} &= h \left( \frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \right), \\ \eta_{k,2} &= h \left( -\frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \right). \end{aligned}$$

It can be verified that  $(\eta_{k,1}, \eta_{k,2})$ ,  $k = 1, 2, \dots$ , are independent bounded random vectors under  $\widehat{\mathbb{E}}$ . Let  $x_k = \sqrt{2 \log \log n_k}$ . By Theorem 3.2, for sufficiently large  $k$  we have

$$\begin{aligned}\widehat{\mathbb{E}}[\eta_{k,1}] &\geq \mathbb{V}\left(\frac{S_{n_k} - S_{n_{k-1}}}{(V_{n_k}^2 - V_{n_{k-1}}^2)^{1/2} \sqrt{2 \log \log n_k}} \geq 1 - \epsilon\right) \\ &\geq \exp\left\{-\frac{1}{2}(1 - \epsilon)x_k^2\right\} \geq ck^{1-\epsilon/2}.\end{aligned}$$

It follows that

$$\sum_{k=1}^{\infty} \widehat{\mathbb{E}}[\eta_{k,1}] = \infty.$$

So, by the Bernstein inequality (4.6) we have

$$\begin{aligned}1 - \mathbb{V}\left(\sum_{k=1}^n \eta_{k,1} > \frac{1}{2} \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]\right) &= \mathcal{V}\left(\sum_{k=1}^n \eta_{k,1} \leq \frac{1}{2} \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]\right) \\ &= \mathcal{V}\left(\sum_{k=1}^n (\eta_{k,1} - \widehat{\mathbb{E}}[\eta_{k,1}]) \geq \frac{1}{2} \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]\right) \\ &\leq \exp\left\{-\frac{(\sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}])^2/4}{8 \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}^2] + 4 \cdot \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]/2}\right\} \\ &\leq \exp\left\{-\frac{1}{40} \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]\right\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.\end{aligned}$$

By the continuity of  $\mathbb{V}$ ,

$$\begin{aligned}\mathbb{V}\left(\sum_{k=1}^{\infty} \eta_{k,1} = \infty\right) &= \mathbb{V}\left(\bigcup_{n=1}^{\infty} \bigcap_{N=n}^{\infty} \left\{\sum_{k=1}^N \eta_{k,1} > \frac{1}{2} \sum_{k=1}^N \widehat{\mathbb{E}}[\eta_{k,1}]\right\}\right) \\ &\geq \limsup_{n \rightarrow \infty} \mathbb{V}\left(\sum_{k=1}^n \eta_{k,1} > \frac{1}{2} \sum_{k=1}^n \widehat{\mathbb{E}}[\eta_{k,1}]\right) = 1.\end{aligned}$$

Similarly,

$$\mathbb{V}\left(\sum_{k=1}^{\infty} \eta_{k,2} = \infty\right) = 1.$$

Now, by the independence of  $\{(\eta_{k,1}, \eta_{k,2})\}$ ,

$$\begin{aligned}
& \mathbb{V} \left( \sum_{k=1}^{\infty} \eta_{k,1} = \infty \text{ and } \sum_{k=1}^{\infty} \eta_{k,2} = \infty \right) \\
&= \lim_{N_1 \rightarrow \infty} \lim_{M_1 \rightarrow \infty} \mathbb{V} \left( \sum_{k=1}^{N_1} \eta_{k,1} \geq M_1 \text{ and } \sum_{k=N_1+1}^{\infty} \eta_{k,2} = \infty \right) \\
&= \lim_{N_2 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \lim_{N_1 \rightarrow \infty} \lim_{M_1 \rightarrow \infty} \mathbb{V} \left( \sum_{k=1}^{N_1} \eta_{k,1} \geq M_1 \text{ and } \sum_{k=N_1+1}^{N_2} \eta_{k,2} \geq M_2 \right) \\
&\geq \lim_{N_2 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \lim_{N_1 \rightarrow \infty} \lim_{M_1 \rightarrow \infty} \widehat{\mathbb{E}} \left[ g \left( \sum_{k=1}^{N_1} \eta_{k,1}/M_1 \right) g \left( \sum_{k=N_1+1}^{N_2} \eta_{k,2}/M_2 \right) \right] \\
&= \lim_{N_2 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \lim_{N_1 \rightarrow \infty} \lim_{M_1 \rightarrow \infty} \widehat{\mathbb{E}} \left[ g \left( \sum_{k=1}^{N_1} \eta_{k,1}/M_1 \right) \right] \widehat{\mathbb{E}} \left[ g \left( \sum_{k=N_1+1}^{N_2} \eta_{k,2}/M_2 \right) \right] \\
&\geq \lim_{N_2 \rightarrow \infty} \lim_{M_2 \rightarrow \infty} \lim_{N_1 \rightarrow \infty} \lim_{M_1 \rightarrow \infty} \mathbb{V} \left( \sum_{k=1}^{N_1} \eta_{k,1} \geq 2M_1 \right) \mathbb{V} \left( \sum_{k=N_1+1}^{N_2} \eta_{k,2} \geq 2M_2 \right) \\
&= \mathbb{V} \left( \sum_{k=1}^{\infty} \eta_{k,1} = \infty \right) \mathbb{V} \left( \sum_{k=1}^{\infty} \eta_{k,2} = \infty \right) = 1,
\end{aligned}$$

where  $g(x)$  is a Lipschitz function with  $I\{x \geq 1\} \geq g(x) \geq I\{x \geq 2\}$ . Combing the above inequality, (4.17) and (4.19) we obtain

$$\begin{aligned}
& \mathbb{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \geq 1 - 2\epsilon \text{ and } \liminf_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \leq -(1 - 2\epsilon) \right) \\
&\geq \mathbb{V} \left( \sum_{k=1}^{\infty} \eta_{k,1} = \infty \text{ and } \sum_{k=1}^{\infty} \eta_{k,2} = \infty \right) = 1.
\end{aligned}$$

By the continuity of  $\mathbb{V}$ , letting  $\epsilon \rightarrow 0$  we obtain (4.13).  $\square$

## 5 Normal random variables

In this section, we show that in (3.2)  $\mathbb{V}$  can be replaced by  $\mathcal{V}$  when the random variables are normal distributed, and so

$$\mathcal{V} \left( \left\{ \frac{S_n}{V_n \sqrt{2 \log \log n}} \right\} \rightarrow [-1, 1] \right) = \mathbb{V} \left( \left\{ \frac{S_n}{V_n \sqrt{2 \log \log n}} \right\} \rightarrow [-1, 1] \right) = 1.$$

Let  $0 < \underline{\sigma} \leq \bar{\sigma} < \infty$  and  $G(\alpha) = \frac{1}{2}(\bar{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-)$ .  $X$  is call a normal  $N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$  distributed random variable (write  $X \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ ) under  $\widehat{\mathbb{E}}$ , if for any bounded



Lipschitz function  $\varphi$ , the function  $u(x, t) = \widehat{\mathbb{E}}[\varphi(x + \sqrt{t}X)]$  ( $x \in \mathbb{R}, t \geq 0$ ) is the unique viscosity solution of the following heat equation:

$$\partial_t u - G(\partial_{xx}^2 u) = 0, \quad u(0, x) = \varphi(x).$$

**Theorem 5.1** *Let  $\{X, X_n; n \geq 1\}$  be a sequence of independent and identically distributed normal random variables with  $X_i \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$  under  $\widehat{\mathbb{E}}$ . Suppose that  $\mathbb{V}$  is continuous. Then*

$$\mathcal{V}\left(\left\{\frac{S_n}{V_n \sqrt{2 \log \log n}}\right\} \rightarrow [-1, 1]\right) = 1. \quad (5.1)$$

To prove Theorem 5.1, we recall the definition of  $G$ -Brownian motion. Let  $C[0, \infty)$  be a function space of continuous functions on  $[0, \infty)$  with the norm  $\|x\| = \sum_{k=1}^{\infty} 2^{-k} \sup_{0 \leq t \leq k} |x(t)|$  and  $C_b(C[0, \infty))$  is the set of bounded continuous functions  $h(x) : C[0, \infty) \rightarrow \mathbb{R}$ . It is showed that there is a sub-linear expectation space  $(\widetilde{\Omega}, \widetilde{\mathcal{H}}, \widetilde{\mathbb{E}})$  with  $\widetilde{\Omega} = C[0, \infty)$  and  $C_b(C[0, \infty)) \subset \widetilde{\mathcal{H}}$  such that  $(\widetilde{\mathcal{H}}, \widetilde{\mathbb{E}}[\|\cdot\|])$  is a Banach space, and the canonical process  $W(t)(\omega) = \omega_t$  ( $\omega \in \widetilde{\Omega}$ ) is a  $G$ -Brownian motion with  $W(1) \sim N(0, [\underline{\sigma}^2, \overline{\sigma}^2])$  under  $\widetilde{\mathbb{E}}$ , i.e., for all  $0 \leq t_1 < \dots < t_n$ ,  $\varphi \in C_{l, lip}(\mathbb{R}^n)$ ,

$$\widetilde{\mathbb{E}}[\varphi(W(t_1), \dots, W(t_{n-1}), W(t_n) - W(t_{n-1}))] = \widetilde{\mathbb{E}}[\psi(W(t_1), \dots, W(t_{n-1}))], \quad (5.2)$$

where  $\psi(x_1, \dots, x_{n-1}) = \widetilde{\mathbb{E}}[\varphi(x_1, \dots, x_{n-1}, \sqrt{t_n - t_{n-1}}W(1))]$  (c.f. Peng (2006, 2008a, 2010), Denis, Hu and Peng (2011)).

We denote a pair of capacities corresponding to the sub-linear expectation  $\widetilde{\mathbb{E}}$  by  $(\widetilde{\mathbb{V}}, \widetilde{\mathbb{V}})$ . Then  $\widetilde{\mathbb{V}}$  and  $\widetilde{\mathbb{V}}$  are continuous.

Denis, Hu and Peng (2011) showed the following representation of the  $G$ -Brownian motion (c.f. Theorem 52).

**Lemma 5.1** *Let  $(\Omega, \mathcal{F}, P)$  be a probability measure space and  $\{B(t)\}_{t \geq 0}$  is a  $P$ -Brownian motion. Then for all bounded continuous function  $\varphi : C_b[0, \infty) \rightarrow \mathbb{R}$ ,*

$$\widetilde{\mathbb{E}}[\varphi(W(\cdot))] = \sup_{\theta \in \Theta} E_P[\varphi(W_{\theta}(\cdot))], \quad W_{\theta}(t) = \int_0^t \theta(s) dB(s),$$

where

$$\Theta = \{\theta : \theta(t) \text{ is } \mathcal{F}_t\text{-adapted process such that } \underline{\sigma} \leq \theta(t) \leq \overline{\sigma}\},$$

$$\mathcal{F}_t = \sigma\{B(s) : 0 \leq s \leq t\} \vee \mathcal{N}, \quad \mathcal{N} \text{ is the collection of } P\text{-null subsets.}$$

**Proof of Theorem 5.1.** By (3.1) and Proposition 2.1 of Griffin and Kuelbs (1989), it is sufficient to show that

$$\mathcal{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \geq 1 \text{ and } \liminf_{n \rightarrow \infty} \frac{S_n}{V_n \sqrt{2 \log \log n}} \leq -1 \right) = 1.$$

Note the sub-additive of  $\mathbb{V}$  and the continuity of  $\mathbb{V}$  and  $\mathcal{V}$ . It is sufficient to show that for all  $\epsilon > 0$ ,

$$\mathcal{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{\epsilon + V_n \sqrt{2 \log \log n}} > 1 - \epsilon \right) = 1. \quad (5.3)$$

Let  $W(t)$  be a  $G$ -Brownian motion on  $(\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})$  with  $W(1) \sim N(0, [\underline{\sigma}^2, \bar{\sigma}^2])$ . Denote  $\tilde{V}_n^2 = \sum_{k=1}^n (W(k) - W(k-1))^2$ . By the continuity of  $\mathcal{V}$  and  $\tilde{\mathcal{V}}$  again,

$$\begin{aligned} & \mathcal{V} \left( \limsup_{n \rightarrow \infty} \frac{S_n}{\epsilon + V_n \sqrt{2 \log \log n}} > 1 - \epsilon \right) \\ &= \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \mathcal{V} \left( \max_{n \leq k \leq N} \frac{S_k}{\epsilon + V_k \sqrt{2 \log \log k}} > 1 - \epsilon \right) \\ &\geq \lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \tilde{\mathcal{V}} \left( \max_{n \leq k \leq N} \frac{W(k)}{\epsilon + \tilde{V}_k \sqrt{2 \log \log k}} > 1 - 2\epsilon \right) \\ &\quad \left( \text{by the fact } (X_1, \dots, X_N) \stackrel{d}{=} (W(1) - W(0), \dots, W(N) - W(N-1)) \right) \\ &= \tilde{\mathcal{V}} \left( \limsup_{n \rightarrow \infty} \frac{W(n)}{\epsilon + \tilde{V}_n \sqrt{2 \log \log n}} > 1 - 2\epsilon \right) \\ &= \inf_{\theta \in \Theta} P \left( \limsup_{n \rightarrow \infty} \frac{W_\theta(n)}{\epsilon + V_\theta(n) \sqrt{2 \log \log n}} > 1 - 2\epsilon \right) \end{aligned}$$

by Lemma 5.1, where  $V_\theta^2(n) = \sum_{k=1}^n (W_\theta(k) - W_\theta(k-1))^2$ . So, it is sufficient to show that for each  $\theta \in \Theta$ ,

$$\limsup_{n \rightarrow \infty} \frac{W_\theta(n)}{V_\theta(n) \sqrt{2 \log \log n}} = 1 \text{ a.s. } P. \quad (5.4)$$

Let  $m_k = (W_\theta(k) - W_\theta(k-1))^2 - \int_{k-1}^k \theta^2(s) ds$ . It is easily seen that  $\{m_k, \mathcal{F}_k\}$  is a sequence of martingale differences with  $E_P[m_k^2 | \mathcal{F}_{k-1}] \leq 4\bar{\sigma}^4$ . By the law of large numbers for martingales,

$$\frac{1}{n} \left( V_\theta^2(n) - \int_0^n \theta^2(s) ds \right) = \frac{1}{n} \sum_{k=1}^n m_k \rightarrow 0 \text{ a.s. } P.$$

It is obvious that  $n\underline{\sigma}^2 \leq \int_0^n \theta^2(s) ds \leq n\bar{\sigma}^2$ . It follows that

$$\frac{V_\theta^2(n)}{\int_0^n \theta^2(s) ds} \rightarrow 1 \text{ a.s. } P.$$

On the other hand, note that  $W_\theta(t) = \int_0^t \theta(s)dB(s)$  is a continuous martingale with quadratic variation process  $\langle W_\theta, W_\theta \rangle(t) = \int_0^t \theta^2(s)ds$ . By the Dambis-Dubins-Schwarz theorem, there is a standard Brownian motion  $B$  under  $P$  such that  $W_\theta(t) = B(\langle W_\theta, W_\theta \rangle_t)$ . So, it is sufficient to show that

$$\limsup_{n \rightarrow \infty} \frac{B(\langle W_\theta, W_\theta \rangle_n)}{\sqrt{2\langle W_\theta, W_\theta \rangle_n \log \log n}} = 1 \quad a.s. \ P. \quad (5.5)$$

Note  $\langle W_\theta, W_\theta \rangle_t \rightarrow \infty$  and is a continuous function of  $t$ . By the law of the iterated logarithm for Brownian motion,

$$\limsup_{t \rightarrow \infty} \frac{B(\langle W_\theta, W_\theta \rangle_t)}{\sqrt{2\langle W_\theta, W_\theta \rangle_t \log \log \langle W_\theta, W_\theta \rangle_t}} = \limsup_{t \rightarrow \infty} \frac{B(t)}{\sqrt{2t \log \log t}} = 1 \quad a.s. \ P,$$

which implies (5.5) by noting that  $\langle W_\theta, W_\theta \rangle_t \approx t$ ,  $\max_{n \leq t \leq n+1} |\langle W_\theta, W_\theta \rangle_t - \langle W_\theta, W_\theta \rangle_n| \leq \bar{\sigma}^2$  and the path properties of a Brownian motion. The proof is now completed.  $\square$

**Remark 5.1** *We conjecture that for all random variables satisfying the conditions Theorem 3.1, (5.1) holds, and*

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \mathcal{V}(S_n \geq x_n V_n) = \lim_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(S_n \geq x_n V_n) = -\frac{1}{2}, \quad (5.6)$$

whenever  $x_n \rightarrow \infty$  and  $x_n = o(\sqrt{n})$  as  $n \rightarrow \infty$ . (5.1) and (5.6) are interesting because they show that the self-normalized law of the iterated logarithm and the self-normalized moderate deviation under the sub-linear expectation are the same as those under the classical linear expectation. The self-normalization eliminates the effect of the non-linearity.

Following the lines of the proofs of (4.13) and Proposition 4.4, it is sufficient to show that there is a  $\underline{b}_n > 0$  such that

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \int_0^\infty \mathcal{V}(e^{\lambda \eta_n} > t) dt = 2\lambda^2 - \lambda, \quad \lambda > 1/2, \quad (5.7)$$

where  $\eta_n = 2\underline{b}_n S_n - (\underline{b}_n V_n)^2$ . With similar arguments as showing Lemma 4.2 we can show that for  $\underline{b}_n = \frac{1}{z_n} \frac{\widehat{\mathbb{E}}[X_1^2 \wedge z_n^2]}{\widehat{\mathcal{E}}[X_1^2 \wedge z_n^2]}$ ,

$$\widehat{\mathcal{E}}[e^{\lambda \eta_n}] = \exp \{ (2\lambda^2 - \lambda)x_n^2 + o(x_n^2) \}, \quad \lambda > 1/2.$$

Unfortunately, we are not able to conclude (5.7) from the above equality because (4.10) is not true for  $\widehat{\mathcal{E}}$ .

## 6 Non-identically distributed random variables

In this section, we consider the independent but not necessarily identically distributed random variables. We give the self-normalized moderate deviation and the self-normalized law of the iterated logarithm similar to those for classical random variables in a probability space, which were established by Jing, Shao and Wang (2003). Suppose that  $\{X_n; n \geq 1\}$  is a sequence of independent random variables on the sub-linear expectation space  $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$  with  $\widehat{\mathbb{E}}[X_k^2] < \infty$ ,  $k = 1, 2, \dots$ . Let  $\overline{B}_n^2 = \sum_{k=1}^n \widehat{\mathbb{E}}[X_k^2]$ ,  $\underline{B}_n^2 = \sum_{k=1}^n \widehat{\mathcal{E}}[X_k^2]$ , and

$$\Delta_{n,x} = \frac{1}{\overline{B}_n^2} \sum_{k=1}^n \widehat{\mathbb{E}} \left[ X_k^2 \left( 1 \wedge \left| \frac{x}{\overline{B}_n} X_k \right| \right) \right].$$

**Theorem 6.1** Suppose  $\widehat{\mathbb{E}}[X_k] \leq 0$ . Let  $q_n = \overline{B}_n^2 / \underline{B}_n^2$ . Then for  $x \geq 2$ ,

$$\mathbb{V}(S_n \geq xV_n) \leq \exp \left\{ -\frac{x^2}{2} + O(1)q_n^3 (\log x + x^2 \Delta_{n,x}) \right\}, \quad (6.1)$$

where  $|O(1)| \leq C$  with  $C$  does not depend on  $x$ .

Further, suppose  $\widehat{\mathbb{E}}[X_k] = \widehat{\mathcal{E}}[X_k] = 0$ ,  $\limsup_{n \rightarrow \infty} \overline{B}_n^2 / \underline{B}_n^2 < \infty$ ,  $x_n \rightarrow \infty$  and

$$x_n^2 \max_{i \leq n} \widehat{\mathbb{E}}[X_i^2] = o(\overline{B}_n^2), \quad \Delta_{n,x_n} \rightarrow 0.$$

Then

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(S_n \geq x_n V_n) = -\frac{1}{2}. \quad (6.2)$$

**Theorem 6.2** Suppose  $\widehat{\mathbb{E}}[X_k] = \widehat{\mathcal{E}}[X_k] = 0$ ,  $\overline{B}_n \rightarrow \infty$ ,  $\limsup_{n \rightarrow \infty} \overline{B}_n^2 / \underline{B}_n^2 < \infty$ ,

$$\max_{i \leq n} \widehat{\mathbb{E}}[X_i^2] = o(\overline{B}_n^2 / \log \log \overline{B}_n)$$

and that

$$\forall \epsilon > 0, \quad \frac{1}{\overline{B}_n^2} \sum_{i=1}^n \widehat{\mathbb{E}} \left[ \left( X_i^2 - \epsilon \overline{B}_n^2 / \log \log \overline{B}_n \right)^+ \right] \rightarrow 0. \quad (6.3)$$

Then

$$\mathcal{V} \left( \limsup_{n \rightarrow \infty} \frac{|S_n|}{V_n \sqrt{2 \log \log \overline{B}_n}} \leq 1 \right) = 1 \quad (6.4)$$

when  $\mathbb{V}$  is countably sub-additive; and

$$\mathbb{V} \left( \left\{ \frac{S_n}{V_n \sqrt{2 \log \log \overline{B}_n}} \right\} \rightarrow [-1, 1] \right) = 1 \quad (6.5)$$

when  $\mathbb{V}$  is continuous.

It is easily seen that the condition (6.3) implies that  $\Delta_{n,x_n} \rightarrow 0$  for  $x_n = (1 \pm \epsilon)\sqrt{2 \log \log \overline{B}_n^2}$  and then (6.2). After having (6.2), Theorem 6.1 can be proved by similar arguments as showing Theorem 3.1 combining with the arguments as in the proof of Theorem 4.1 of Jing, Shao and Wang (2003). We omitted the details here. The proof of Theorem 6.1 will be completed via four propositions.

**Proposition 6.1** *Suppose  $\widehat{\mathbb{E}}[X_k] \leq 0$ . We have*

$$\mathbb{V}\left(S_n \geq xV_n, V_n^2 \geq 9\overline{B}_n^2\right) \leq 2 \exp\left\{-x^2 + O(1)x^2\Delta_{n,x}\right\}. \quad (6.6)$$

**Proof.**  $b = b_x = x/\overline{B}_n$ ,  $\widehat{S}_n = \sum_{i=1}^n X_i \wedge (A_0/b)$  where  $A_0$  is an absolute constant to be determined later. Observe that

$$\begin{aligned} & \mathbb{V}\left(S_n \geq xV_n, V_n^2 \geq 9\overline{B}_n^2\right) \\ & \leq \mathbb{V}\left(\widehat{S}_n \geq xV_n/2, V_n^2 \geq 9\overline{B}_n^2\right) + \mathbb{V}\left(\sum_{i=1}^n (X_i - A_0/b)^+ \geq xV_n/2\right) \\ & \leq \mathbb{V}\left(\widehat{S}_n \geq \frac{3}{2}x\overline{B}_n\right) + \mathbb{V}\left(\sum_{i=1}^n I\{bX_i > A_0\} \geq \frac{x^2}{4}\right). \end{aligned}$$

Note  $e^s \leq 1 + s + \frac{s^2}{2} + \frac{e^s}{6}(s^3 \vee 0)$ . We have

$$\begin{aligned} & \widehat{\mathbb{E}}\left[\exp\left\{\frac{3}{2}(bX_i) \wedge A_0\right\}\right] \\ & \leq \widehat{\mathbb{E}}\left[1 + \frac{3}{2}(bX_i) \wedge A_0 + \frac{9}{8}((bX_i) \wedge A_0)^2 + \frac{27e^{3A_0/2}}{48}|bX_i|^3 \wedge A_0^3\right] \\ & \leq 1 + \frac{3}{2}\widehat{\mathbb{E}}[bX_i] + \frac{9}{8}\widehat{\mathbb{E}}[(bX_i)^2] + \frac{27e^{3A_0/2}}{48}A_0^3\widehat{\mathbb{E}}[|bX_i|^3 \wedge 1] \\ & \leq \exp\left\{\frac{9}{8}b^2\widehat{\mathbb{E}}[X_i^2] + \frac{27e^{3A_0/2}}{48}A_0^3\widehat{\mathbb{E}}[|bX_i|^3 \wedge 1]\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} & \mathbb{V}\left(\widehat{S}_n \geq \frac{3}{2}x\overline{B}_n\right) \leq \exp\left\{-\frac{9}{4}x^2\right\} \widehat{\mathbb{E}}\left[\exp\left\{\frac{3}{2}b\widehat{S}_n\right\}\right] \\ & \leq \exp\left\{-\frac{9}{4}x^2\right\} \prod_{i=1}^n \widehat{\mathbb{E}}\left[\exp\left\{\frac{3}{2}(bX_i) \wedge A_0\right\}\right] \\ & \leq \exp\left\{-\frac{9}{4}x^2\right\} \exp\left\{\frac{9}{4}b^2\overline{B}_n^2 + \frac{27e^{3A_0/2}}{48}A_0^3x^2\Delta_{n,x}\right\} \\ & = \exp\left\{-\frac{9}{4}x^2 + \frac{27e^{3A_0/2}}{48}A_0^3x^2\Delta_{n,x}\right\}. \end{aligned}$$

On the other hand, let  $h(x)$  be a Lipschitz function such that  $I\{x > A_0\} \leq h(x) \leq I\{x > A_0/2\}$ . Then

$$\begin{aligned}
& \mathbb{V} \left( \sum_{i=1}^n I\{bX_i > A_0\} \geq \frac{x^2}{4} \right) \leq \mathbb{V} \left( \sum_{i=1}^n h(bX_i) \geq \frac{x^2}{4} \right) \\
& \leq \exp \left\{ -t \frac{x^2}{4} \right\} \prod_{i=1}^n \widehat{\mathbb{E}} [\exp \{t h(bX_i)\}] \\
& \leq \exp \left\{ -t \frac{x^2}{4} \right\} \prod_{i=1}^n \widehat{\mathbb{E}}^* [1 + e^t I\{x > A_0/2\}] \\
& \leq \exp \left\{ -t \frac{x^2}{4} \right\} \prod_{i=1}^n \left( 1 + e^t \frac{4}{A_0^2} \widehat{\mathbb{E}}[(bX_i)^2] \right) \\
& \leq \exp \left\{ -t \frac{x^2}{4} + e^t \frac{4}{A_0^2} b^2 \overline{B}_n^2 \right\} \\
& \leq \exp \left\{ -t \frac{x^2}{4} + \frac{4e^t x^2}{A_0^2} \right\} \leq \exp \{-x^2\}
\end{aligned}$$

if we choose  $t = 5$  and  $A_0 = 120$ . The proof is completed.  $\square$

Let  $\lambda > 0$  and  $\theta > 0$  be two real numbers. Define  $f(s) = e^{\lambda s - \theta s^2}$  as in (4.3). Then

$$\begin{aligned}
f(s) &= 1 + \lambda s + \left( \frac{\lambda^2}{2} - \theta \right) s^2 + g(s) \quad \text{with} \\
|g(s)| &\leq \left( \frac{1}{2} \theta^{3/2} e^{\frac{\lambda^2}{4\theta}} + 2e^{-\frac{3}{2}} \theta e^{\frac{\lambda^2}{4\theta}} \right) (s^2 \wedge |s|^3).
\end{aligned}$$

Similar to Lemma 4.1, we have the following lemma.

**Lemma 6.1** *Suppose that  $b$  is a positive number and  $\xi$  is a random variable. Then*

$$\begin{aligned}
& \widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} \\
& \leq 1 + b^2 \left( \frac{\lambda^2}{2} - \theta \right)^+ \widehat{\mathbb{E}} [\xi^2] - b^2 \left( \frac{\lambda^2}{2} - \theta \right)^- \widehat{\mathcal{E}} [\xi^2] + O_{\lambda, \theta} \left( \widehat{\mathbb{E}}[|b\xi|^3 \wedge (b\xi)^2] \right)
\end{aligned}$$

if  $\widehat{\mathbb{E}}[\xi] \leq 0$ , and

$$\begin{aligned}
& \widehat{\mathbb{E}} \exp \{ \lambda(b\xi) - \theta(b\xi)^2 \} \\
& \geq 1 + b^2 \left( \frac{\lambda^2}{2} - \theta \right)^+ \widehat{\mathbb{E}} [\xi^2] - b^2 \left( \frac{\lambda^2}{2} - \theta \right)^- \widehat{\mathcal{E}} [\xi^2] + O_{\lambda, \theta} \left( \widehat{\mathbb{E}}[|b\xi|^3 \wedge (b\xi)^2] \right)
\end{aligned}$$

if  $\widehat{\mathcal{E}}[\xi] \geq 0$ , where  $|O_{\lambda, \theta}| \leq \frac{1}{2} \theta^{3/2} e^{\frac{\lambda^2}{4\theta}} + 2e^{-\frac{3}{2}} \theta e^{\frac{\lambda^2}{4\theta}}$ .

Hence we have the following lemma similar to Lemma 4.2.

**Lemma 6.2** Suppose  $\widehat{\mathbb{E}}[X_i^2] < \infty$ ,  $i \geq 1$ ,  $x \geq 2$ .

(a) Suppose  $\widehat{\mathbb{E}}[X_i] \leq 0$  ( $i \geq 1$ ). Let  $b = b_n = x/\overline{B}_n$  if  $\frac{\lambda^2}{2} - \theta \geq 0$ . Then

$$\widehat{\mathbb{E}} \exp \{ \lambda(bS_n) - \theta(bV_n)^2 \} \leq \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) x^2 + O_{\lambda, \theta} x^2 \Delta_{n,x} \right\}.$$

(b) Suppose  $\widehat{\mathbb{E}}[X_i] \leq 0$  ( $i \geq 1$ ). Let  $b = b_n = x/\underline{B}_n$  if  $\frac{\lambda^2}{2} - \theta < 0$ . Then

$$\widehat{\mathbb{E}} \exp \{ \lambda(bS_n) - \theta(bV_n)^2 \} \leq \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) x^2 + O_{\lambda, \theta} q_n^3 x^2 \Delta_{n,x} \right\}.$$

(c) Suppose  $\widehat{\mathbb{E}}[X_i] \geq 0$  ( $i \geq 1$ ),  $\overline{B}_n^2 \rightarrow \infty$ ,  $\frac{\lambda^2}{2} - \theta > 0$ ,  $x_n \geq 2$ ,

$$x_n^2 \max_{i \leq n} \widehat{\mathbb{E}}[X_i^2] = o(\overline{B}_n^2).$$

Let  $b = b_n = x_n/\overline{B}_n$ . Then

$$\widehat{\mathbb{E}} \exp \{ \lambda(bS_n) - \theta(bV_n)^2 \} \geq \exp \left\{ \left( \frac{\lambda^2}{2} - \theta \right) x_n^2 + O_{\lambda, \theta} x_n^2 \Delta_{n,x_n} \right\}.$$

Here  $|O_{\lambda, \theta}| \leq C(\theta^{3/2} e^{\frac{\lambda^2}{4\theta}} + \theta e^{\frac{\lambda^2}{4\theta}})$ .

**Proposition 6.2** Suppose  $\widehat{\mathbb{E}}[X_i] \leq 0$ ,  $i \geq 1$ , and  $0 < \delta \leq \frac{1}{4} \frac{B_n^2}{\overline{B}_n^2} \leq \frac{1}{4}$ . For  $x \geq 2$ ,

$$\mathbb{V} \left( S_n \geq xV_n, V_n^2 \leq \delta \overline{B}_n^2 \right) \leq \exp \left\{ -2x^2 + O(1)x^2 q_n^3 \Delta_{n,x} \right\}.$$

**Proof.** Let  $b = b_n = x/\underline{B}_n$ . By Lemma 6.2 (b) we have for  $\lambda = 2$ ,

$$\begin{aligned} & \mathbb{V} \left( S_n \geq xV_n, V_n^2 \leq \delta \overline{B}_n^2 \right) \\ &= \mathbb{V} \left( bS_n \geq x\sqrt{(bV_n)^2}, (bV_n)^2 \leq \delta \frac{\overline{B}_n^2}{\underline{B}_n^2} x^2 \right) \\ &\leq \mathbb{V} \left( bS_n \geq x\sqrt{(bV_n)^2}, (bV_n)^2 \leq \frac{1}{4} x^2 \right) \\ &\leq \mathbb{V} (bS_n - 2(bV_n)^2 \geq 0) \leq \widehat{\mathbb{E}} \exp \{ \lambda(bS_n - 2(bV_n)^2) \} \\ &\leq \exp \left\{ \left( \frac{\lambda^2}{2} - 2\lambda \right) x^2 + O_{\lambda, 2\lambda} q_n^3 x^2 \Delta_{n,x} \right\} \\ &= \exp \left\{ -2x^2 + O(1)q_n^3 x^2 \Delta_{n,x} \right\}. \quad \square \end{aligned}$$

**Proposition 6.3** Suppose  $\widehat{\mathbb{E}}[X_i] \leq 0$ ,  $i \geq 1$ , and  $\delta = \frac{1}{4} \frac{B_n^2}{\overline{B}_n^2}$ . For  $x \geq 2$ ,

$$\mathbb{V} \left( S_n \geq xV_n, \delta \overline{B}_n^2 \leq V_n^2 \leq 9\overline{B}_n^2 \right) \leq \exp \left\{ -\frac{x^2}{2} + O(1)q_n^{3/2} (\log x + x^2 \Delta_{n,x}) \right\}.$$

**Proof.** Let  $1 < \theta < 2$ ,  $b = b_n = \frac{x}{B_n}$ . Similar to the proof of Proposition 4.2, by Lemma 6.2 (a) we have

$$\begin{aligned}
& \mathbb{V} \left( S_n \geq xV_n, \delta \overline{B}_n^2 \leq V_n^2 \leq 9\overline{B}_n^2 \right) \\
& \leq \mathbb{V} \left( 2\frac{x}{V_n}S_n - \left(\frac{x}{V_n}V_n\right)^2 \geq x^2, \frac{1}{3}b_n \leq \frac{x}{V_n} \leq \delta^{-1/2}b_n \right) \\
& \leq \mathbb{V} \left( \sup_{3^{-1} \leq a \leq \delta^{-1/2}} (2ab_n S_n - (ab_n V_n)^2) \geq x^2 \right) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \mathbb{V} \left( \sup_{\theta^j \leq a \leq \theta^{j+1}} (2ab_n S_n - (ab_n V_n)^2) \geq x^2 \right) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \mathbb{V} (2\theta^{j+1}b_n S_n - (\theta^j b_n V_n)^2 \geq x^2) \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \exp\left\{-\frac{x^2}{2}\right\} \widehat{\mathbb{E}} \exp \left\{ \theta^{j+1}b_n S_n - \frac{1}{2}\theta^{2j}(b_n V_n)^2 \right\} \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \exp\left\{-\frac{x^2}{2}\right\} \exp \left\{ \left( \frac{1}{2}\theta^{2j+2} - \frac{1}{2}\theta^{2j} \right) x^2 + O_{\theta^{j+1}, \theta^{2j}/2} x^2 \Delta_{n,x} \right\} \\
& \leq \sum_{\left[\frac{\log 3^{-1}}{\log \theta}\right] \leq j \leq \frac{\log \delta^{-1/2}}{\log \theta}} \exp \left\{ -\frac{x^2}{2} + \frac{x^2}{2}(\theta^2 - 1)\theta^{2j} + C\theta^{3j}e^{\theta/2}x^2\Delta_{n,x} \right\} \\
& \leq \left( \frac{\log \delta^{-1/2} + \log 3}{\log \theta} + 1 \right) \exp \left\{ -\frac{x^2}{2} + \frac{1}{2}(\theta^2 - 1)\delta^{-1}x^2 + C\delta^{-3/2}e^{\theta/2}x^2\Delta_{n,x} \right\}.
\end{aligned}$$

Let  $\theta^2 = 1 + \delta x^{-2}$ . It is easily seen that

$$\frac{\log \delta^{-1/2} + \log 3}{\log \theta} + 1 \leq \exp \left\{ O(1)(\log \delta^{-1} + \log x) \right\}.$$

It follows that

$$\mathbb{V} \left( S_n \geq xV_n, \delta \overline{B}_n^2 \leq V_n^2 \leq 9\overline{B}_n^2 \right) \leq \exp \left\{ -\frac{x^2}{2} + O(1)\delta^{-3/2} (\log x + x^2 \Delta_{n,x}) \right\}.$$

The proof is completed.  $\square$

Now, (6.1) and the upper bound of (6.2) follows from Propositions 6.1-6.3 immediately. As for the lower bound of (6.2), we let  $b = b_n = x_n/\overline{B}_n$ ,  $\eta_n = 2b_n S_n - (b_n V_n)^2$ . Then by Lemma 6.2 (a) and (c),

$$\lim_{n \rightarrow \infty} x_n^{-2} \ln \widehat{\mathbb{E}} \exp \{ \lambda \eta_n \} = 2\lambda^2 - \lambda, \quad \lambda > 1/2,$$

which implies the following proposition similar to Proposition 4.4.



**Proposition 6.4** For any  $0 < \beta < 1$ ,

$$\liminf_{n \rightarrow \infty} x_n^{-2} \ln \mathbb{V}(\eta_n \geq (1 + \beta)x_n^2) \geq -\frac{(1 + \beta/4)^2}{2}. \quad (6.7)$$

Then, the lower bound of (6.2) follows by noting

$$\mathbb{V}(S_n \geq x_n V_n) \geq \mathbb{V}(2b_n S_n - (b_n V_n)^2 \geq x_n^2).$$

The proofs are now completed.

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